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STANDARD 3-COMPONENTS OF TYPE Sp(6, 2)

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ABSTRACT. It is shown that if G is a finite simple group with a standard 3-component of type Sp(6, 2) and G satisfies certain 2-local and 3-local conditions then either G is isomorphic to Sp(8, 2) or G is isomorphic to $F_4(2)$.

1. Introduction. As part of their program to classify finite simple groups of characteristic 2-type, Gorenstein and Lyons **[9]** introduced a set of Standard form problems for odd primes. In a standard form problem for the prime 3, one is concerned with groups having the following properties.

HYPOTHESIS A. G is a finite group with an element b of order 3 such that $C = C(b)$ satisfies the following conditions.

(A1) C has a quasisimple normal subgroup L;

(A2) $C(L)$ has cyclic Sylow 3-subgroups;

(A3) $m_{2,3}(G) = m_{2,3}(C)$;

(A4) $\langle b \rangle$ is not weakly closed in C with respect to G;

(A5) if $B^* \in \beta_{\mathrm{Max}}(C; 3)$, then B^* acts nontrivially on some 2-subgroup of C.

Here $\beta_{\mathrm{Max}}(C; 3)$ is the collection of elementary abelian 3-subgroups of C of maximal rank with respect to lying in a 2-local subgroup. The main result of this paper is the following theorem.

THEOREM A. *Let G be a finite group of characteristic* 2-*type with $O^3(G) = G$ and $O_{3'}(G) = \langle 1 \rangle$. If G satisfies Hypothesis* A *with L isomorphic to* Sp(6, 2), *then G is isomorphic to* Sp(8, 2) *or to* $F_4(2)$.

Theorem A provides the solution to a standard form problem required by Aschbacher in his work on the classification of characteristic 2-type simple groups G satisfying $e(G) = 3$. In general, we say that L is a standard 3-component of the characteristic 2-type simple group G if the conditions of Hypothesis A are satisfied. Although our definition of standard form differs from that given by Gorenstein and Lyons **[9]**, the conditions of Hypothesis A follow from their result for most Chevalley groups defined over $GF(2)$. If L has the property that $m_{2,3}(L) < m_3(L)$, then it is easy to verify that (A3) and (A5) are equivalent to the following hypothesis:

(A3)′ $m_{2,3}(G) = m_{2,3}(L) + 1$.

In particular, (A3)′ holds when L is isomorphic to Sp(6, 2).

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In [4] and [5] the authors classified groups with a standard 3-component isomorphic to $L_n(2)$, $n \geqslant 5$, and in [6] the first author and R. Solomon classified groups with a standard 3-component isomorphic to $\mathrm{Sp}(2n, 2)$, $n > 4$. The proof of Theorem A follows the same outline as these papers. We begin the analysis by determining the fusion of b in C. Two fusion patterns emerge corresponding to the cases where G is isomorphic to $\mathrm{Sp}(8, 2)$ or $F_4(2)$ respectively. In the $\mathrm{Sp}(8, 2)$ case we are able to show that G contains an involution t which centralizes L and such that $C(t)$ is isomorphic to the centralizer of a transvection in $\mathrm{Sp}(8, 2)$. A generators and relations argument using Curtis' theorem [2] then yields a subgroup Y of G with $Y \cong \mathrm{Sp}(8, 2)$ and $C_Y(t) = C(t)$. The identification of Y with G is made via Holt's theorem [11]. It should be noted that the generators and relations argument used in [7] breaks down when L is isomorphic to $\mathrm{Sp}(6, 2)$.

In the $F_4(2)$ case, we first determine partial information about the structure of the centralizers of two involutions of G. These subgroups correspond to the maximal parabolic subgroups of $F_4(2)$ obtained by eliminating a node at either end of the Dynkin diagram of F_4. This information, although incomplete, is sufficient, when coupled with our knowledge of certain 3-local subgroups of G, to allow us to apply Curtis' theorem to construct a subgroup Y of G isomorphic to $F_4(2)$. As before, Holt's theorem is then used to prove that $Y = G$.

2. Properties of certain Chevalley groups. In this section, we enumerate the properties of certain Chevalley groups required for the proof of Theorem A.

Let (,) be a nonsingular symplectic form on a vector space V of dimension $2n + 2$ over $GF(2)$ and let $\mathrm{Sp}(V)$ be the group of the nonsingular linear transformations of V which preserve (,). Choose a basis $\{e_0, f_0, \dots, e_n, f_n\}$ for V so that

$$(e_i, e_j) = (f_i, f_j) = 0, \qquad (e_i, f_j) = \delta_{ij},$$

$0 \leqslant i, j \leqslant n$. Let $V_i = \langle e_i, f_i \rangle$, $0 \leqslant i \leqslant n$, so that $V = V_0 \perp V_1 \perp \cdots \perp V_n$. For $A \subseteq \{0, 1, \dots, n\}$ let $V_A = \langle V_i \colon i \in A \rangle$. Since $C_{\mathrm{Sp}(V)}(V_{A'})$ acts naturally as $\mathrm{Sp}(V_A)$ on V_A, we may identify $C_{\mathrm{Sp}(V)}(V_{A'})$ with $\mathrm{Sp}(V_A)$. Using this convention, it follows that if $A \cap B = \varnothing$, then $[\mathrm{Sp}(V_A), \mathrm{Sp}(V_B)] = 1$. Let K be the symmetric group on $\{0, 1, \dots, n\}$. Then we may think of K as a subgroup of $\mathrm{Sp}(V)$ with the action of $\sigma \in K$ on V given by

$$e_i\sigma = e_{i\sigma}, \qquad f_j\sigma = f_{j\sigma}$$

for $0 \leqslant i, j \leqslant n$. Note that $\mathrm{Sp}(V_A)^\sigma = \mathrm{Sp}(V_{A\sigma})$ for $A \subseteq \{0, 1, \dots, n\}$. In particular, if $\mathrm{Sp}(V_i) = \langle b_i, t_i \rangle \cong \Sigma_3$ with $\langle b_i \rangle = O_3(\mathrm{Sp}(V_i))$ and $B = \langle b_0, b_1, \dots, b_n \rangle$, then $N(B) = (\langle b_0, t_0 \rangle \times \cdots \times \langle b_n, t_n \rangle)K \cong \Sigma_3 \wr \Sigma_{n+1}$. We summarize important properties of $\mathrm{Sp}(V)$ in the following result.

(2.1) The following conditions hold in $\mathrm{Sp}(V)$:

(i) B contains representatives of all the conjugacy classes of elements of order 3 of $\mathrm{Sp}(V)$.

(ii) B is the unique $E_{3^{n+1}}$-subgroup of $N(B)$. Hence $N(B)$ contains a Sylow 3-subgroup of $\mathrm{Sp}(V)$ and $m_3(\mathrm{Sp}(V)) = n + 1$.

(iii) $m_{2,3}(\mathrm{Sp}(V)) = n$.

(iv) Let z be an involution of $\mathrm{Sp}(V)$ and assume that $n > 2$. Then some conjugate of z centralizes b_0 or b_0b_1.

(v) Let t be a transvection of $\mathrm{Sp}(V)$. Then $C(t) = TL$ where $T = O_2(C(t)) \cong E_{2^{2n+1}}$ and L acts as $O(2n+1, 2)$ on T.

(vi) $N(\langle b_i \rangle) = \langle b_i, t_i \rangle \times \mathrm{Sp}(V_{i'})$, where $i' = \{i\}'$ and $\mathrm{Sp}(V_{i'})$ acts naturally on $V_{i'} = C_V(b_i)$, $i = 0, 1, \ldots, n$.

PROOF. See [6, §2].

(2.2) All subgroups of $SL(V)$ isomorphic to $\mathrm{Sp}(V)$ are conjugate to $\mathrm{Sp}(V)$.

PROOF. This is presumably well known, but for the convenience of the reader, we supply an argument. The proof is by induction on n. The result is clear if $n = 0$ or 1, so assume $n \geqslant 2$. Let H be a subgroup of $SL(V)$ isomorphic to $\mathrm{Sp}(V)$. Since $\mathrm{Sp}(V)$ contains a Sylow 3-subgroup P of $SL(V)$, we may assume that $P \subseteq \mathrm{Sp}(V) \cap H$. Without loss, it may be further assumed by (2.1ii) that $B = J(P)$. As $N_{\mathrm{Sp}(V)}(B) = N_{SL(V)}(B)$, it follows that $N_{\mathrm{Sp}(V)}(B) = \langle b_i, t_i\colon 0 \leqslant i \leqslant n \rangle K \subseteq H \cap \mathrm{Sp}(V)$. If β is an element of order 3 of H such that $C_H(\beta) \cong Z_3 \times \mathrm{Sp}(2n, 2)$, then $n \geqslant 2$ implies that $\beta =_H b_0$. Thus $O^{2'}(C_H(b_0))$ is isomorphic to $\mathrm{Sp}(2n, 2)$, centralizes V_0, and acts faithfully on $V_{0'}$. By induction, there exists $x \in C_{SL(V)}(V_0)$ such that $x^{-1}(O^{2'}(C_H(b_0)))x = \mathrm{Sp}(V_{0'})$. Hence $C_{\mathrm{Sp}(V)}(b_0) \subseteq H^x \cap \mathrm{Sp}(V)$. Since $B \subseteq C_{\mathrm{Sp}(V)}(b_0)$, we also have $\langle C_{\mathrm{Sp}(V)}(b_0), N_{\mathrm{Sp}(V)}(B) \rangle \subseteq H^x \cap \mathrm{Sp}(V)$. This implies that if $A \subseteq \{0, 1, \ldots, n\}$, $|A| = 2$, then $\mathrm{Sp}(V_A) \subseteq H^x \cap \mathrm{Sp}(V)$. But these subgroups generate $\mathrm{Sp}(V)$ by [6, Lemma 2.7], hence $H^x = \mathrm{Sp}(V)$.

(2.3) Let $L \cong \mathrm{Sp}(2n, 2)$ act faithfully on $T \cong E_{2^{2n+1}}$ with $C_T(L) = \langle t \rangle \cong Z_2$. Then either $T = \langle t \rangle \times [T, L]$ or L acts as $O(2n+1, 2)$ on T.

PROOF. It suffices to show that $X = C_{\mathrm{Aut}(T)}(t)$ has precisely two conjugacy classes of $\mathrm{Sp}(2n, 2)$ subgroups. Let $U = O_2(X)$. Then $U \cong E_{2^{2n}}$ and $X = U \cdot \mathrm{Aut}(U)$. By (2.3), $\mathrm{Aut}(U)$ has one conjugacy class of $\mathrm{Sp}(2n, 2)$ subgroups. Hence the problem is further reduced to showing that $U \cdot L$ has two conjugacy classes of complements. The assertion is a consequence of a result of Pollatsek [13], so the proof is finished.

We now discuss the relationship between the geometric and Chevalley representations of $\mathrm{Sp}(V)$. Let $\{w_0, w_1, \ldots, w_n\}$ be an orthonormal basis of $\mathbf{R}^{n+1}$. We define a root system Φ of type C_{n+1} as follows:

$$\Phi = \{\pm w_i \pm w_j, 2w_i\colon 0 \leqslant i \neq j \leqslant n\}.$$

(2.4) $\mathrm{Sp}(V)$ is generated by the involutions $\{U_\alpha(1)\colon \alpha \in \Phi\}$. The matrices of $U_\alpha(1)$, $\alpha \in \Phi$, relative to $\{e_0, f_0, \ldots, e_n, f_n\}$ satisfy the following;

(i) $\mathrm{mat}(U_\alpha(1)) = \mathrm{mat}(U_{-\alpha}(1))^T$.

(ii) $U_{w_i \pm w_j}(1)$ acts as the identity on V_k, $k \notin \{i, j\}$. With respect to $\{e_i, f_i, e_j, f_j\}$,

$$\mathrm{mat}(U_{w_i - w_j}(1)) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\mathrm{mat}(U_{w_i + w_j}(1)) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(iii) $U_{2w_i}(1)$ acts as the identity on V_k, for $k \neq i$. With respect to $\{e_i, f_i\}$,

$$\mathrm{mat}\big(U_{2w_i}(1)\big) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Also, we have the Chevalley commutator formula

(iv) $[U_\alpha(1), U_\beta(1)] = \prod U_{i\alpha+j\beta}(C_{ij;\alpha\beta})$, where i, j are positive integers such that $i\alpha + j\beta \in \Phi$ and $C_{ij;\alpha\beta}$ are the Chevalley structure constants for the Lie algebra of type C_{n+1}.

A fundamental system for Φ is $\{p_0, p_1, \ldots, p_n\}$ where $p_0 = 2w_0$ and $p_i = w_i - w_{i+1}$, $1 \leq i \leq n$. Let $A_{ij} = \Phi \cap \langle p_i, p_j\rangle$ and set $A = \bigcup A_{ij}$, $0 \leq i, j \leq n$. The next result is due to Curtis [**2**] and Steinberg [**14**] and gives a convenient presentation for $\mathrm{Sp}(V)$.

(2.5) A presentation for $\mathrm{Sp}(V)$ ($\dim(V) \geq 8$) is obtained from the generators $\{U_\alpha(1): \alpha \in A\}$ and the relations

(i) $U_\alpha(1)^2 = 1$ and

(ii) $[U_\alpha(1), U_\beta(1)] = \prod U_{i\alpha+j\beta}(C_{ij;\alpha\beta})$,

where α, β are independent roots in the same A_{ij}.

In (2.4), we described how $\mathrm{Sp}(V)$ may be identified with the Chevalley group $C_{n+1}(2)$. However, it will be necessary in §6 to use the isomorphism between $\mathrm{Sp}(V)$ and the group $B_{n+1}(2)$. In this case, we may define the root system Φ^* dual to Φ by setting $r^* = 2r/(r, r)$ and

$$\Phi^* = \{r^*: r \in \Phi\}.$$

Note that $r^* = r$ if $r = \pm w_i \pm w_j$, $i \neq j$, whereas $2w_i^* = w_i$. Thus Φ^* has type B_{n+1} and r^* is long if and only if r is short. It then follows from a result of Ree that (2.4) may be restated by replacing Φ with Φ^* and r with r^* to obtain the isomorphism between $\mathrm{Sp}(V)$ and $B_{n+1}(2)$.

(2.6) Definition. Let G be a Chevalley group of normal type defined over $GF(2)$ with root system Σ. An $SL(2, 2)$-subgroup K of G is said to be a root $SL(2, 2)$-subgroup if K is conjugate to $\langle U_{\pm\alpha}\rangle$ for some $\alpha \in \Sigma$. Here $U_\alpha = \langle U_\alpha(1)\rangle$, $\alpha \in \Sigma$. If Σ has roots of different lengths, then K is said to be a long (short) root $SL(2, 2)$-subgroup.

The next result is important for the construction of $F_4(2)$ in §6 and may be of some independent interest.

(2.7) The following is true.

(i) All $SL(3, 2)$-subgroups of $SL(n, 2)$ $(n > 3)$ which contain a root $SL(2, 2)$-subgroup are conjugate.

(ii) Regard $\mathrm{Sp}(V)$ as $C_{n+1}(2)$ $(n > 2)$. Then all $SL(3, 2)$-subgroups of $\mathrm{Sp}(V)$ which contain a short root $SL(2, 2)$-subgroup are conjugate.

Proof. (i) Let U be a vector space of dimension n over $GF(2)$ and let H be an $SL(3, 2)$-subgroup of $SL(U)$ which contains a root $SL(2, 2)$-subgroup. It suffices to show that $U = U_1 \oplus U_2$ where $U_1 = [U, H]$ has dimension 3 and $U_2 = C_U(H)$. By [**5**, (6.3)] and assumption, $H = \langle \tau_1, \tau_2, \tau_3\rangle$ where each τ_i is a root involution of $SL(U)$. Since root involutions of $SL(U)$ act as transvections on U, $\dim([U, \tau_i]) = 1$, $1 \leq i \leq 3$. In particular, $\dim(C_U(\tau_i)) = n - 1$, $1 \leq i \leq 3$, and it follows that

$U_2 = C_U(H) = \bigcap_{i=1}^3 C_U(\tau_i)$ has dimension $n - 3$. Set $U_1 = \langle [U, \tau_i]: 1 \leqslant i \leqslant 3 \rangle$. Then U_1 has dimension 3 and is invariant under H as $[U_1, \tau_i] \subseteq U_1$, $1 \leqslant i \leqslant 3$. Thus $U = U_1 \oplus U_2$ is the required decomposition of U.

(ii) Assume first that $\dim(V) = 6$. In the previous notation for $\mathrm{Sp}(V)$, we have $n = 2$ and $\Sigma = \{\pm w_i \pm w_j,\ 0 \leqslant i \neq j \leqslant 2\}$ is a subsystem of Φ of type A_3 with fundamental system $\{w_1 - w_2, w_0 - w_1, w_1 + w_2\}$. Hence $J = \langle U_r(1): r \in \Sigma \rangle \cong A_3(2) \cong SL(4, 2)$. Furthermore, the element n_{w_2} of the Weyl group of $\mathrm{Sp}(V)$ acts as the graph automorphism on J so that $J^* = \langle J, n_{w_2} \rangle = N_{\mathrm{Sp}(V)}(J) \cong \mathrm{Aut}(J)$. Consider the permutation representation of $\mathrm{Sp}(V)$ on the set Ω of the 36 $\mathrm{Sp}(V)$-conjugates of J^*. We shall show that if H is an $SL(3, 2)$-subgroup of $\mathrm{Sp}(V)$ containing a root $SL(2, 2)$-subgroup, then H is conjugate to a subgroup of J. To do this, it suffices to show that H fixes a point of Ω. Without loss, we may assume that $\langle U_{\pm(w_1 - w_2)} \rangle \subseteq H \cap J$. Using the permutation character of $\mathrm{Sp}(V)$ acting on Ω given in [**8**], we then see that an involution of H fixes 12 points of Ω. Moreover, an F_{21} subgroup of H acting on Ω has orbits of lengths 1, 7, 7 and 21. The only possible set of orbit lengths of H compatible with this information is then easily seen to be $\{1, 7, 7, 21\}$. Thus H is conjugate to a subgroup of J. But then H is an $SL(3, 2)$-subgroup of J containing a root $SL(2, 2)$-subgroup of J; hence by (i), H is conjugate in J to $\langle U_{\pm(w_1 - w_2)}, U_{\pm(w_1 - w_0)} \rangle$. As H is arbitrary, the result follows.

Now assume that $\dim(V) > 6$. Let $H = \langle \tau_1, \tau_2, \tau_3 \rangle$ where each τ_i is a short root involution of $\mathrm{Sp}(V)$. Let $U = \Sigma_{i=1}^3 [V, \tau_i]$. Since $\dim([V, \tau_i]) = 2$ by (2.4ii), $\dim(U) \leqslant 6$. Evidently $[V, H] \subseteq U$, hence H normalizes U and acts trivially on V/U. The irreducible $GF(2)$-modules for $SL(3, 2)$ have dimensions 1, 3, 3 and 8. Also if β is an element of order 3 of a short root $SL(2, 2)$-subgroup of H, then $[V, \beta]$ is the orthogonal sum of two hyperbolic planes. An easy argument then yields $\dim(U) = 6$, and $U \cap U^{\perp} = \{0\}$. As H normalizes $U^{\perp}$ and $[H, U^{\perp}] \subseteq U \cap U^{\perp} = \{0\}$, H centralizes $U^{\perp}$. But U is conjugate under $\mathrm{Sp}(V)$ to $V_0 \perp V_1 \perp V_2$ and accordingly, H is conjugate in $\mathrm{Sp}(V)$ to a subgroup of $\mathrm{Sp}(V_{012})$. We may now appeal to the case $\dim(V) = 6$ to complete the argument.

Finally, we complete our discussion of the symplectic groups by proving the following result which is crucial for our construction of $F_4(2)$.

(2.8) Let K be a short (resp. long) root $SL(2, 2)$-subgroup of $C_3(2)$ (resp. $B_3(2)$). Then K commutes with exactly one long and one short root $SL(2, 2)$-subgroup.

PROOF. We prove the result only for $C_3(2)$. For convenience, let $K = \langle U_{\pm(w_0 - w_1)} \rangle$. By (2.4ii), we compute that $N(K) = K \times \langle U_{\pm(w_0 + w_1)} \rangle \times \langle U_{\pm 2w_2} \rangle$. The result now follows by inspection.

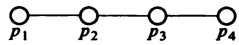
We turn our attention to $F_4(2)$ and enumerate those properties required for the proof of Theorem A. Let $\{w_1, w_2, w_3, w_4\}$ be an orthonormal basis of $\mathbf{R}^4$. We define a root system Δ of type F_4 as follows:

$$\Delta = \left\{ \begin{array}{l} \pm w_i \pm w_j\colon 1 \leqslant i \neq j \leqslant 4 \\ \pm w_j\colon 1 \leqslant i \leqslant 4 \\ \dfrac{1}{2}(\pm w_1 \pm w_2 \pm w_3 \pm w_4) \end{array} \right\}.$$

A fundamental system for Δ is given by $\{p_1, p_2, p_3, p_4\}$ where

$$p_1 = \frac{1}{2}(w_1 - w_2 - w_3 - w_4), \quad p_2 = w_4, \quad p_3 = w_3 - w_4 \quad \text{and} \quad p_4 = w_2 - w_3.$$

This gives rise to the following diagram:



$F_4(2)$ is generated by the involutions $\{U_\alpha(1)\colon \alpha \in \Delta\}$. If α and β are independent roots of Δ, then we have the Chevalley commutator formula

$$[U_\alpha(1), U_\beta(1)] = U_{i\alpha + j\beta}(C_{ij;\alpha\beta}),$$

where i, j are positive integers such that $i\alpha + j\beta \in \Delta$ and $C_{ij;\alpha\beta}$ are the structure constants for the Lie algebra of type F_4.

In constructing $F_4(2)$, it is convenient for us to reformulate Curtis' theorem in the following way.

(2.9) Let Y be a group generated by $SL(2, 2)$-subgroups $\{K_i\colon 1 \leq i \leq 4\}$. Set $Y_1 = \langle K_1, K_2\rangle$, $Y_2 = \langle K_2, K_3, K_4\rangle$ and assume that there exist monomorphisms π_i: $Y_i \to F_4(2)$, $i = 1, 2$, such that $\pi_i(K_j) = \langle U_{\pm p_j}\rangle$, $1 \leq j \leq 4$. Further assume that $[K_j, K_l] = 1$, $|j - l| > 1$. Then there exists an epimorphism π: $Y \to F_4(2)$ with $\ker \pi \subseteq Z(Y)$ and $\pi(K_j) = \langle U_{\pm p_j}\rangle$, $1 \leq j \leq 4$.

PROOF. Let $\Delta_1 = \Delta \cap \langle p_1, p_2\rangle$ and $\Delta_2 = \langle p_2, p_3, p_4\rangle$. Then Δ_1 has type A_2 and Δ_2 has type B_3. Since $\pi_i(Y_i) = \langle U_\alpha\colon \alpha \in \Delta_i\rangle$, it follows that $Y_1 \cong SL(3, 2)$ and $Y_2 \cong \mathrm{Sp}(6, 2)$. We claim that $Y_1 \cap Y_2 = K_2$. Otherwise, $Y_1 \cap Y_2 \cong \Sigma_4$ or $SL(3, 2)$ and an easy argument using [**1**, (6.2)] implies that $Y_1 \cap Y_2$ contains an involution $k_1 \in K_1$. But then $[k_1, \langle K_3, K_4\rangle] = 1$ and this contradicts the fact (see [**1**]) that no involution of $\mathrm{Sp}(6, 2)$ centralizes an $SL(3, 2)$-subgroup.

We want to show that π_1 may be adjusted so that $\pi_1|K_1 = \pi_2|K_1$. Let $\tilde{U}^*_\alpha = \pi_2^{-1}(U_\alpha)$, $r \in \Delta_2$. If $U = \tilde{U}^*_{p_2} \neq \tilde{U}_{\pm p_2}$, then by [**5**, (6.3)], $\langle K_1, U\rangle = \langle K_1, K_2\rangle$. Since $\tilde{U}^*_{-p_3}$ centralizes $\langle K_1, U\rangle$, we then have that $[\tilde{U}^*_{-p_2}, \tilde{U}^*_{-p_3}] = 1$ which is impossible. Similarly, $\tilde{U}_{-p_2} \neq \tilde{U}^*_{\pm p_2}$ yields a contradiction and we conclude that $\{\tilde{U}_{p_2}, \tilde{U}_{-p_2}\} = \{\tilde{U}^*_{p_2}, \tilde{U}^*_{-p_2}\}$. There exists an outer automorphism of $\langle K_1, K_2\rangle$ which leaves K_i invariant and interchanges $\tilde{U}_{p_i}$ with $\tilde{U}_{-p_i}$, $i = 1, 2$. Preceding π_1 by this automorphism if necessary, we may assume that $\tilde{U}_{p_2} = \tilde{U}^*_{p_2}$ and $\tilde{U}_{-p_2} = \tilde{U}^*_{-p_2}$. Therefore $\pi_1|K_2 = \pi_2|K_2$. In particular, this implies that the map $\pi_1 \cup \pi_2$ is well defined on $Y_1 \cup Y_2$.

Let $A_{ij} = \Delta \cap \langle p_i, p_j\rangle$ and set $A = \bigcup A_{ij}$, $1 \leq i, j \leq 4$. Observe that $A_{ij} = \{\pm p_i, p_j\}$ if $|i - j| > 1$. In particular, $A \subseteq \Delta_1 \cup \Delta_2$, hence for each $\alpha \in A$, we may define $\tilde{U}_\alpha(1) = (\pi_1 \cup \pi_2)^{-1}(U_\alpha(1))$. Clearly $Y = \langle \tilde{U}_\alpha(1)\colon \alpha \in A\rangle$. Suppose that α and β are independent roots which are linear combinations of the fundamental roots p_i and p_j. If $|i - j| > 1$, then $[\tilde{U}_\alpha(1), \tilde{U}_\beta(1)] = 1$. On the other hand, if $|i - j| \leq 1$, then $\langle \tilde{U}_\alpha(1), \tilde{U}_\beta(1)\rangle \subseteq Y_i$, $i = 1$ or 2. In either case, we have the Chevalley commutator formula

$$[\tilde{U}_\alpha(1), \tilde{U}_\beta(1)] = \prod \tilde{U}_{i\alpha + j\beta}(C_{ij:\alpha\beta}).$$

Therefore, by a result of Curtis [**2**], there exists an epimorphism π: $Y \to F_4(2)$ such that $\pi|Y_i = \pi_i$, $i = 1, 2$, and $\ker \pi \subseteq Z(Y)$.

The next result summarizes some properties of the 2-structure of $F_4(2)$.

(2.10) The following hold.

(i) $F_4(2)$ has 4 classes of involutions. Representatives of these classes are $U_r(1)$, $U_s(1)$, $U_r(1)U_s(1)$, and $U_\alpha(1)U_\beta(1)$. Here $r = 2p_1 + 4p_2 + 3p_3 + 2p_4$, $s = 2p_1 + 3p_2 + 2p_3 + p_4$, $\alpha = p_1 + 2p_2 + 2p_3 + p_4$ and $\beta = 2p_1 + 4p_2 + 2p_3 + p_4$.

(ii) If $U = \langle U_\gamma \colon \gamma \in \Delta^+ \rangle$, then U is a Sylow 2-subgroup of $F_4(2)$, $|U| = 2^{24}$ and $Z(U) = \langle U_r, U_s \rangle$ (with r, s as in (i)).

(iii) $C_{F_4(2)}(U_r(1)) \cong C_{F_4(2)}(U_s(1))$ is an extension of $E_{2^6} \times X_{2^9}^+$ by Sp(6, 2).

(iv) Each involution of $F_4(2)$ is centralized by a short and a long root $SL(2, 2)$-subgroup.

(v) $[\mathrm{Aut}(F_4(2)) \colon F_4(2)] = 2$. All involutions of $\mathrm{Aut}(F_4(2)) \setminus F_4(2)$ are conjugate to the graph automorphism τ. Also $C_{F_4(2)}(\tau) \cong {}^2F_4(2)$.

PROOF. See Guterman **[10]** and Aschbacher-Seitz **[1]**.

In dealing with $F_4(2)$ it is helpful to keep in mind that the graph automorphism τ induces a symmetry between subgroups of $F_4(2)$ generated by short root involutions and those generated by long root involutions.

3. Proof of Theorem A: fusion of b in C. In this section and for the remainder of the paper, we shall assume that G satisfies the hypotheses of Theorem A. Thus G is a group of characteristic 2-type which satisfies Hypothesis A with L isomorphic to Sp(6, 2). In addition, $O_{3'}(G) = 1$ and $G = O^3(G)$.

With respect to notation given for Sp(V) in §2, let $n = 3$ and $V' = V_1 \perp V_2 \perp V_3$. Then $C_{\mathrm{Sp}(V)}(V_0)$ acts naturally as Sp(V') on V'. Hence we may set $L = C_{\mathrm{Sp}(V)}(V_0)$ and use the notation for L that it inherits as a subgroup of Sp(V). Accordingly, let $B_1 = \langle b_1, b_2, b_3 \rangle \cong E_{3^3}$ so that $N_L(B_1) = (\langle b_1, t_1 \rangle \times \langle b_2, t_2 \rangle \times \langle b_3, t_3 \rangle)K_1$, where $K_1 = K \cap L \cong \Sigma_3$. Set $B = \langle b, B_1 \rangle$. The main result of this section is the following.

PROPOSITION 3.1. *Let $X = \mathrm{Aut}_G(B)$. Then $\langle b \rangle^G \cap \mathcal{E}_1(B) = \langle b \rangle^X$ and one of the following occurs*:

(i) $\langle b \rangle^X = \{\langle b \rangle, \langle b_1 \rangle, \langle b_2 \rangle, \langle b_3 \rangle\}$ *and X is equal to the monomial group M on B with respect to* $\{b, b_1, b_2, b_3\}$, *or*

(ii) *X contains M, $O_2(X) \cong Q_8 * Q_8$ and either*

(a) $X/O_2(X) \cong \Sigma_3 \times \Sigma_3$, $\langle b \rangle^X = \langle b \rangle^M \cup \langle bb_1b_2b_3 \rangle^M$, *or*

(b) $X/O_2(X) \cong \Sigma_3 \wr Z_2$, $\langle b \rangle^X = \langle b \rangle^M \cup \langle bb_1 \rangle^M \cup \langle bb_1b_2b_3 \rangle^M$.

We shall eventually show that G is isomorphic to Sp(8, 2) or to $F_4(2)$ according to whether (i) or (iia) of Proposition 3.1 holds respectively. When case (iib) holds, we shall prove that G contains a subgroup isomorphic to $\mathrm{Aut}(F_4(2))$ and then derive a contradiction to the assumption that G has characteristic 2-type.

The proof of Proposition 3.1 is presented in a sequence of lemmas. By (2.1), L satisfies (i)–(iv) of Hypothesis I of **[5]**, hence part of the fusion analysis of **[5]** can be applied to the present situation.

(3.2) The following hold.

(i) $C = O_{3'}(C) \times \langle b \rangle \times L$ where $O_{3'}(C)$ has odd order and $m_{2,3}(G) = m_{2,3}(C) = 3$.

(ii) $b^G \cap B = b^X$.

(iii) $b^G \cap B_1 \neq \varnothing$.

PROOF. Parts (i) and (ii) are easy consequences of the arguments presented in [6, (3.2), (3.3)] and [5, (4.17)]. To prove (iii), we first apply the remark at the end of the proof of Proposition 1 of [5] to obtain $b^G \cap L \neq \varnothing$. Since B_1 contains representatives of all the classes of elements of order 3 of L by (2.1i), it then follows from (ii) that $b^X \cap B_1 \neq \varnothing$.

It follows from (3.2iii) that b is strongly real in G. Since L admits no nontrivial outer automorphisms, we may choose an involution $t_0 \in C(L)$ such that $\langle b, t_0\rangle \cong \Sigma_3$. To conform with the notation of [5], set $b = b_0$. Also let $F_1 = \langle t_1, t_2, t_3\rangle$ so that $N_L(B_1) = B_1F_1K_1$ and choose the involutions t_i, so that K_1 permutes the set $\{t_1, t_2, t_3\}$. Thus F_1K_1 acts as the monomial group on B_1 with respect to $\{b_1, b_2, b_3\}$, and $O^{2'}(C_{N(B)}(t_0)) = \langle t_0\rangle \times B_1F_1K_1$. Since $C_L(B_1) = B_1$, it follows from (3.2i) that $C(B)$ has odd order. Hence $C_X(t_0) = \langle t_0\rangle \times F_1K_1$. As $b_0^X \cap B_1 \neq \varnothing$, we may apply Proposition 5.1 [5] to establish the following.

(3.3) One of the following holds.

(i) Proposition 3.1 is true.

(ii) X contains a subgroup isomorphic to $Z_2 \times \Sigma_6$ with index at most 2. If $\Delta(\beta)$ is the orbit of $N_L(B_1)$ on $\mathcal{E}_1(B)$ containing $\langle \beta\rangle$, then either

(a) $\langle b_0\rangle^X = \Delta(b_0) \cup \Delta(b_1b_2) \cup \Delta(b_0b_1b_2b_3)$, or

(b) $\langle b_0\rangle^X = \Delta(b_0) \cup \Delta(b_1) \cup \Delta(b_1b_2) \cup \Delta(b_0b_1b_2) \cup \Delta(b_0b_1b_2b_3)$.

It is evident from the preceding result that Proposition 3.1 will be proved once case (ii) of (3.3) has been eliminated. Before beginning this argument, we require the following elementary result.

(3.4) Assume that a π-group P acts on a π'-group Q with $Q_0 = C_Q(P)$. Then $Z(Q_0) \subseteq Z(N_Q(Z(Q_0)))$. In particular, if $Q_0 \trianglelefteq P$, then $Z(Q_0) \subseteq Z(Q)$.

PROOF. It suffices to assume that $Z(Q_0) \trianglelefteq Q$. Then $[Q, Z(Q_0), P] \subseteq [Q_0, P] = 1$, so $[[P, Q], Z(Q_0)] = 1$ by the 3 subgroups lemma. Therefore $Q = Q_0[P, Q]$ centralizes $Z(Q_0)$.

At this point we assume that case (ii) of (3.3) occurs. Let $T = O_2(C(t_0))$. As G has characteristic 2-type, L acts faithfully on T. We shall eventually show that this leads to a contradiction.

Evidently $N_L(B_1)$ has 3 orbits on $\mathcal{E}_2(B_1)$ with representatives $A_1 = \langle b_1, b_2\rangle$, $A_2 = \langle b_1b_2, b_3\rangle$ and $A_3 = \langle b_1b_2, b_2b_3\rangle$. Since $|\mathcal{E}_1(A_i) \cap \langle b_0\rangle^G|$ is distinct for $i = 1, 2, 3$, it follows that no two of A_1, A_2 and A_3 are conjugate in G. Furthermore, it is easy to see that $A_1 =_{N(B)} \langle b_0, b_1b_2^2\rangle$. In fact, let $(b_1b_2)^x = b_0$ for $x \in N(B)$. Then $\langle b_0\rangle \subseteq A_i^x \subseteq B$, $i = 1, 2, 3$. As $|\mathcal{E}_1(A) \cap \langle b_0\rangle^G| = |\mathcal{E}_1(A_i^x) \cap \langle b_0\rangle^G|$ the possibilities for A_i^x are easily determined from the information given in (a) or (b) of case (ii). We thus have by inspection that

$$O^{\{2,3\}'}(C(A_1)) \cong E_9 \times \Sigma_3 \times \Sigma_3,$$

$$O^{\{2,3\}'}(C(A_2)) \cong E_9 \times \mathrm{Sp}(4, 2),$$

and

$$O^{\{2,3\}'}(C(A_3)) \cong Z_3 \times GU_3(2).$$

As $B_1 \subseteq C(t_0)$, it is easy to find $S_i = O_2(C(A_i) \cap C(t_0))$ from the above information. Thus $S_1 = \langle t_0 \rangle$, $S_2 \cong E_8$ and $S_3 \cong Q_8$. Note that $C_T(A_i) \hookrightarrow S_i$, $i = 1, 2, 3$, so $C_T(A_1) = \langle t_0 \rangle$ and $N_L(B_1)$ acts faithfully on all nontrivial $N_L(B_1)$-factors of $T/\langle t_0 \rangle$.

We shall use the following lemma concerning the action of $\Sigma_3 \wr \Sigma_3 \cong N_L(B_1)$ on 2 groups to analyze the structure of T.

(3.5) Let $H = \Sigma_3 \wr \Sigma_3$ and let $A = \langle a_1, a_2, a_3 \rangle = O_3(H)$. Also, let $\{a_1, a_2, a_3\}$ be a basis for A on which H/A acts monomially. Suppose that H acts faithfully and irreducibly on a vector space V over $GF(2)$. Then $V = \langle C_V(A_0)^H \rangle$ for some hyperplane A_0 of A and $\dim V = 6$, 8, or 12. The dimensions of the fixed points of the elements of A acting on V are given in the table below.

$\dim V$	$\dim C_V(a_1)$	$\dim C_V(a_1a_2)$	$\dim C_V(a_1a_2a_3)$	A_0
6	4	2	0	$\langle a_1, a_2 \rangle$
8	0	4	2	$\langle a_1a_2, a_2a_3 \rangle$
12	4	2	6	$\langle a_1, a_2a_3 \rangle$

PROOF. By Clifford's theorem V is the direct sum of (nontrivial) irreducible A-modules which are transitively permuted by H. Each nontrivial, irreducible A-module has dimension 2 and is centralized by a hyperplane A_0 of A. It is not difficult to verify that H has three orbits on $\mathcal{E}_2(A)$ with representatives $\langle a_1, a_2 \rangle$, $\langle a_1a_2, a_2a_3 \rangle$ and $\langle a_1, a_2a_3 \rangle$. These orbits have lengths 3, 4, and 6 respectively and thus correspond to modules V of dimensions 6, 8, and 12. The entries in the table follow by counting $|\langle a \rangle \cap \mathcal{E}_1(A_0)^H|$ for appropriate $a \in A$.

(3.6) $C_T(A_2) = \langle t_0 \rangle$.

PROOF. Assume the contrary. Then $C_T(A_2) \hookrightarrow S_2 \cong E_8$ implies that $C_T(A_2) = S_2$. Let T^* be a chief L-section of T such that $C_{T^*}(A_2) \cong E_4$.

We claim that $C_{T^*}(A_3) = 1$. If the claim is false then $C_{T^*}(A_3) \cong E_4$ because $C_T(A_3) \hookrightarrow S_3 \cong Q_8$. In fact T^* covers $C_T(A_i)/\langle t_0 \rangle$, $i = 1, 2, 3$, so that $T^* = T/\langle t_0 \rangle$. We must have that $C_T(A_3) = S_3 \cong Q_8$, so that T is nonabelian. The irreducible action of L on T^* forces $T' = \langle t_0 \rangle = Z(T)$. Thus $S_2 \trianglelefteq T$ and (3.4) implies that $S_2 \leqslant Z(T)$, a contradiction, establishing the claim.

By the claim and (3.5), we must have $T^* \cong E_{2^{12}}$ and $C_{T^*}(b_1b_2) \cong E_4$. But b_1b_2 belongs to an $SL(3, 2)$-subgroup K of L because b_1b_2 belongs to a short root $SL(2, 2)$-subgroup of L by §2. The nontrivial K-sections of T^* have dimension 3 or 8 and b_1b_2 has fixed points on all K-sections of T^*. This contradicts the first sentence of this paragraph.

As $C_T(A_2) = C_T(A_1) = \langle t_0 \rangle$ and $T \neq \langle t_0 \rangle$, we must have $C_T(A_3) \neq \langle t_0 \rangle$. Thus $C_T(A_3) = Q_8$ and T is nonabelian. It follows easily from (3.5) that $N_L(B_1)$ acts irreducibly on $T/\langle t_0 \rangle$ and that $|T/\langle t_0 \rangle| = 2^8$. This implies that T is extra-special of type + so that $N(T)/T \hookrightarrow O^+(8, 2)$. It follows from (3.4) and (3.5) that $C_T(b_1b_2) \cong Q_8 * Q_8$. Hence $C_{C(t_0)}(b_1b_2) \hookrightarrow Z_3 \times \Sigma_3 \times O^+(4, 2)$. In particular, if S is a 2-subgroup of $C_{C(t_0)}(b_1b_2)$ satisfying $[S, B_1] = S$, then $S \subseteq T$. Thus $[S_2, B_1] \subseteq T$ and $S_2 = \langle t_0 \rangle \times [S_2, B_1] \subseteq T$. But A_2 centralizes $S_2 \cong E_8$. This contradicts (3.6).

We have shown that case (ii) of (3.3) does not occur. This completes the proof of Proposition 3.1.

4. Proof of Theorem A: The Sp(8, 2) **case.** In this section we show that if case (i) of Proposition 3.1 holds, then G is isomorphic to Sp(8, 2).

Setting $b = b_0$, we have by assumption that $N(B)$ acts as the monomial group on B with respect to $\{b_0, b_1, b_2, b_3\}$. Therefore, we may set $N(B) = O_{3'}(N(B))BFK$ where $F = \langle t_0, t_1, t_2, t_3\rangle \cong E_{16}$, $K \cong \Sigma_4$, $BF = \times_{i=0}^{3}\langle b_i, t_i\rangle$, and K permutes the set $\{t_0, t_1, t_2, t_3\}$. Let S be the collection of proper subsets of $\{0, 1, 2, 3\}$. If $A \in S$, let $B_A = \langle b_i: i \in A\rangle$, $L_A = O^{2'}(C(B_{A'}))$ and $F_A = \langle t_i: i \in A\rangle$. This is not to be confused with the notation $B_1 = \langle b_1, b_2, b_3\rangle$ and $F_1 = \langle t_1, t_2, t_3\rangle$. Since K may be viewed as the group of permutations of $\{0, 1, 2, 3\}$, it follows that for $\sigma \in K$, $B_A^\sigma = B_{A\sigma}$, $L_A^\sigma = L_{A\sigma}$ and $F_A^\sigma = F_{A\sigma}$.

(4.1) The following hold.

(i) If $A \in S$, then $L_A \cong \mathrm{Sp}(2|A|, 2)$, $B \cap L_A = B_A$, and $F \cap L_A = F_A$.

(ii) $[L_i, L_A] = 1$ if $i \notin A$.

(iii) t_j is a transvection of L_A for $j \in A$.

PROOF. To prove (i), if $0 \notin A$, then the result is an easy consequence of the properties of $L = O^{2'}(C(b_0))$. Otherwise, choose $\sigma \in K$ such that $0 \notin A\sigma$. Conjugating $L_{A\sigma}$, $B_{A\sigma}$ and $F_{A\sigma}$ by σ^{-1} then yields the result in this case as well.

In case (ii), choose $\sigma \in K$ so that $i\sigma = 0$. Then $[L_0, L_{A\sigma}] = 1$, hence conjugating by σ^{-1} gives $[L_i, L_A] = 1$.

Finally, (iii) is true if $0 \notin A$ and we may argue as in (i) and (ii) if $0 \in A$.

(4.2) $C(b_1b_2) = O_{3'}(C(b_1b_2))(\langle b_1, b_2\rangle\langle\delta\rangle \times L_{03})$ where $O_{3'}(C(b_1b_2))$ has odd order and δ is the transposition (1, 2) of K. Therefore, if $T \in И_{C(b_1b_2)}(B_1; 2)$, then $[\langle b_1, b_2\rangle, T] = 1$.

PROOF. Let $\Gamma = C(b_1b_2)$. Observe that δ normalizes $L_{03} = O^{2'}(C(\langle b_1, b_2\rangle)) \cong$ Sp(4, 2) and centralizes $\langle b_0, b_3\rangle$. Hence δ centralizes L_{03}. Evidently, $N_\Gamma(\langle b_1, b_2\rangle) = O_{3'}(C(\langle b_1, b_2\rangle))(\langle b_1, b_2\rangle\langle\delta\rangle \times L_{03})$. Also, $m_{2,3}(\Gamma) = m_{2,3}(G) = 3$ implies that $И_\Gamma(B; 2) = \{1\}$. Let $\bar{\Gamma} = \Gamma/O_{3'}(\Gamma)$. Since $N_{\bar{\Gamma}}(\langle\bar{b}_1\rangle) = \overline{N_\Gamma(\langle b_1, b_2\rangle)}$, the first part will be proved once we have shown that $\langle\bar{b}_1\rangle \trianglelefteq \bar{\Gamma}$. This in turn will be accomplished by verifying that the hypotheses of Proposition 2 of [6] hold in $\bar{\Gamma}$.

First observe that $\langle\bar{b}_1, \bar{\delta}\rangle \cong \Sigma_3$ and $N_{\bar{\Gamma}}(\langle\bar{b}_1\rangle) = O_{3'}(N_{\bar{\Gamma}}(\langle\bar{b}_1\rangle))(\langle\bar{b}_1, \bar{\delta}\rangle \times \bar{L}_{03})$. Secondly, $\langle\bar{b}_1\rangle^{\bar{\Gamma}} \cap \mathfrak{S}_1(N_{\bar{\Gamma}}(\langle\bar{b}_1\rangle)) = \langle\bar{b}_1\rangle$, for otherwise, we would have $\langle b_1, b_1b_2\rangle =_G \langle b_1b_2, \beta\rangle$, for some $\beta \in B \setminus \langle b_1, b_2\rangle$. But this is clearly impossible because $\langle b_1, b_1b_2\rangle$ contains two elements of $\langle b_0\rangle^G \cap \mathfrak{S}_1(B) = \{\langle b_0\rangle, \langle b_1\rangle, \langle b_2\rangle, \langle b_3\rangle\}$. Thirdly, $N_\Gamma(\langle b_0, b_1b_2\rangle) = O_{3'}(C(b_0))(\langle b_0, t_0\rangle \times C_L(b_1b_2))$ with $C_L(b_1b_2) = \langle b_1, b_2\rangle\langle\delta\rangle \times \langle b_3, t_3\rangle$. Hence

$$N_{\bar{\Gamma}}(\langle\bar{b}_0\rangle) = \overline{N_\Gamma(\langle b_0, b_1b_2\rangle)} = O_{3'}(C_{\bar{\Gamma}}(\bar{b}_0))(\langle\bar{b}_1, \bar{\delta}\rangle \times N_{\overline{L_{03}}}(\langle\bar{b}_0\rangle)).$$

Similarly for $N_{\bar{\Gamma}}(\langle\bar{b}_3\rangle)$. Finally, $И_\Gamma(B; 2) = \{1\}$ implies that $И_{\bar{\Gamma}}(\bar{B}; 2) = \{1\}$ as well. Therefore the hypotheses of Proposition 2 [6] hold and we have that $\langle\bar{b}_1\rangle \trianglelefteq \bar{\Gamma}$ as required.

The second statement follows directly from the first.

(4.3) $C(t_0) = TL$ where $T = O_2(C(t_0)) \cong E_{2^7}$ and L acts as $O_7(7, 2)$ on T.

PROOF. Set $T = O_2(C(t_0))$. We consider the action of L and its subgroups on T. As $C_T(b_1b_2) \in И_{C(b_1b_2)}(B_1; 2)$, it follows from (4.2) that $C_T(b_1b_2) \subseteq C_T(b_1) \cap C_T(b_2)$. Similarly $C_T(b_1b_2^{-1}) \subseteq C_T(b_1) \cap C_T(b_2)$. Thus the action of $\langle b_1, b_2\rangle$ on T yields that $T = C_T(b_1)C_T(b_2)$.

By (4.1iii), t_0 is a transvection of $L_{023} = O^{2'}(C(b_1))$. Since $C_T(b_1) \subseteq O_2(C_{L_{023}}(t_0))$ and $C_T(b_1)$ is normalized by $\langle b_2, b_3\rangle \subseteq C_{L_{023}}(t_0)$, it follows from (2.1v) that $C_T(b_1) = O_2(C_{L_{023}}(t_0)) \cong E_{2^5}$. By the same reasoning, $C_T(b_2) \cong E_{2^5}$. From the structure of $C_{L_{023}}(t_0) = C_T(b_1)L_{23}$, we see that $C_T(b_1) \cap C_T(b_2) \cong E_{2^3}$. Thus $T = C_T(b_1)C_T(b_2)$ has order 2^7.

Since L acts irreducibly on $T/\langle t_0\rangle$ either $T \cong E_{2^7}$ or T is extra-special. The latter is impossible as $C_T(b_1) \cong E_{2^5}$. Hence $T \cong E_{2^7}$. Let $X = C(t_0)/T$. Then X acts faithfully on $\bar{T} = T/\langle t_0\rangle$. Since $C_{\bar{T}}(b_1) \cong E_{2^4}$ and $N_X(\langle b_1\rangle) = N_L(\langle b_1\rangle) = \langle b_1, t_1\rangle \times L_{23}$, t_1 acts as a transvection on $\bar{T}$. Thus $L \subseteq \langle t_1^X\rangle$ and $\langle t_1^X\rangle$ is an irreducible subgroup of $\mathrm{Aut}(\bar{T})$ generated by transvections. By a result of McLaughlin **[12]**, either $L = \langle t_1^X\rangle$ or $\langle t_1^X\rangle = \mathrm{Aut}(\bar{T})$. The latter is incompatible with the structure of $N_X(\langle b_1\rangle)$ and we conclude that $X = L$. Therefore $C(t_0) = TL$.

It remains to verify that L acts as $O(7, 2)$ on T. If not, then by (2.3), L acts decomposably on T and hence $t_0 \notin C(t_0)'$. However $O^{2'}(C(b_1) \cap C(t_0)) = C_T(b_1)L_{23}$ and L_{23} acts indecomposably on $C_T(b_1)$. This contradicts $t_0 \notin C(t_0)'$ and the result is proved.

The next result is an easy corollary to (4.3).

(4.4) $O_{3'}(C(b_0)) = 1$.

PROOF. It follows from (4.3) that $И_{C(t_0)}(\langle b_1, b_2, b_3\rangle; 3') = \{O_2(C(t_0))\}$. If σ is the transposition (0, 1) of K, then conjugating $C(t_0)$ by σ gives $И_{C(t_0)}(\langle b_0, b_2, b_3\rangle; 3') = \{O_2(C(t_1))\}$. As $t_1 \in L$, $O_{3'}(C(b_0))$ centralizes both $\langle t_1\rangle$ and $\langle b_0, b_2, b_3\rangle$. Hence $O_{3'}(C(b_0)) \subseteq O_2(C(t_1))$. But $O_{3'}(C(b_0))$ has odd order by (3.2i), so $O_{3'}(C(b_0)) = 1$.

Let $Y = \langle C(t_0), L_{012}\rangle$. Our goal is to show that $Y \cong \mathrm{Sp}(8, 2)$. Using the notation of §2, let $V = V_0 \perp V_1 \perp V_2 \perp V_3$ where V_i has as a basis the hyperbolic pair $\{e_i, f_i\}$, $0 \leqslant i \leqslant 3$. Recall that if $A \subseteq \{0, 1, 2, 3\}$ then $\mathrm{Sp}(V_A) = C_{\mathrm{Sp}(V)}(V_{A'})$ acts naturally on V_A. Also, in terms of the Chevalley presentation of $\mathrm{Sp}(V)$, if $\Phi_A = \Phi \cap \langle w_i\colon i \in A\rangle$, then $\mathrm{Sp}(V_A) = \langle U_\alpha(1)\colon \alpha \in \Phi_A\rangle$. Denote the stabilizer of a vector $v \in V^\#$ by $\mathrm{Sp}(V)_v$. Then

$$\mathrm{Sp}(V)_{f_0} = C\big(U_{2w_0}(1)\big) = O_2(\mathrm{Sp}(V)_{f_0})\mathrm{Sp}(V_{123}),$$

where $O_2(\mathrm{Sp}(V)_{f_0}) \cong E_{2^7}$ and $\mathrm{Sp}(V_{123})$ acts as $O_7(7, 2)$ on $O_2(\mathrm{Sp}(V)_{f_0})$.

By (4.3), there exists an isomorphism $\mathrm{Sp}(V)_{f_0} \to C(t_0)$. This isomorphism may be chosen so that $\mathrm{Sp}(V_A) \to L_A$ for $A \subseteq \{1, 2, 3\}$. In particular, $\langle U_{\pm 2w_i}\rangle \to \langle b_i, t_i\rangle$, $1 \leqslant i \leqslant 3$. We shall show that the homomorphism $\mathrm{Sp}(V)_{f_0} \to C(t_0)$ may be extended to a homomorphism $\mathrm{Sp}(V) \to Y$. This will be done in several stages. Denote the map $\mathrm{Sp}(V)_{f_0} \to C(t_0)$ by $x \to \hat{x}$.

(4.5) $\mathrm{Sp}(V)_{f_0} \to C(t_0)$ may be extended to a map $\mathrm{Sp}(V)_{f_0} \cup \mathrm{Sp}(V_{01}) \cup \mathrm{Sp}(V_{02}) \to Y$ such that $\mathrm{Sp}(V_{0i}) \to Y$ is a homomorphism, $i = 1, 2$. Furthermore, $\mathrm{Sp}(V_{0i}) \to L_{0i}$, $i = 1, 2$, and $\mathrm{Sp}(V_0) \to L_0$.

PROOF. Observe that

$$L_{01} \cap C(t_0) = \langle \hat{U}_{2w_0}, \hat{U}_{w_0 \pm w_1} \rangle \cdot \langle \hat{U}_{\pm 2w_1} \rangle,$$

where $L_{01} \cap C(t_0) \cong Z_2 \times \Sigma_4$ and $O_2((L_{01} \cap C(t_0))') = \langle \hat{U}_{2w_0}(1)\hat{U}_{w_0+w_1}(1), \hat{U}_{2w_0}(1)\hat{U}_{w_0-w_1}(1) \rangle$. Construct a graph Γ (resp. $\hat{\Gamma}$) on the conjugates of $U_{2w_0}(1)$ in $\mathrm{Sp}(V_{01})$ (resp. conjugates of $\hat{U}_{2w_0}(1)$ in L_{01}) by connecting two involutions if they generate a Σ_3 subgroup. We then have the subgraph of Γ:

$$\underset{U_{2w_0}(1)}{\circ} \text{———} \underset{U_{-2w_0}(1)}{\circ} \text{———} \underset{U_{2w_0}(1)U_{2w_1}(1)U_{w_0+w_1}(1)}{\circ} \text{———} \underset{U_{-2w_1}(1)}{\circ} \text{———} \underset{U_{2w_1}(1)}{\circ}$$

These involutions together with the indicated relations provide a presentation of $\mathrm{Sp}(V_{01}) \cong \Sigma_6$.

Now let $\tau = t_0^{b_0}$. Then $\langle \tau, \hat{U}_{2w_0}(1) \rangle = L_0$ and $\langle \tau, \hat{U}_{2w_0}(1) \rangle$ commutes with $\langle \hat{U}_{\pm 2w_1} \rangle = L_1$. Since $L_{01} \cong \Sigma_6$, $\hat{U}_{2w_0}(1) =_{L_{01}} \hat{U}_{2w_1}(1)$ and we may identify the L_{01} conjugates of $\hat{U}_{2w_0}(1)$ with the transpositions of L_{01}. In particular, $\{\hat{U}_{2w_0}(1), \hat{U}_{2w_1}(1)\hat{U}_{w_0+w_1}(1), \hat{U}_{2w_1}(1)\}$ is a maximal set of commuting transpositions. It is easy to see that $\langle \tau, \hat{U}_{2w_0}(1)\hat{U}_{2w_1}(1)\hat{U}_{w_0+w_1}(1) \rangle \cong \Sigma_3$ as $\langle \tau, \hat{U}_{2w_0}(1) \rangle \cong \Sigma_3$ and τ commutes with $\hat{U}_{2w_1}(1)$. Therefore we have the subgraph of $\hat{\Gamma}$:

$$\underset{\hat{U}_{2w_0}(1)}{\circ} \text{———} \underset{\tau}{\circ} \text{———} \underset{\hat{U}_{2w_0}(1)\hat{U}_{2w_1}(1)\hat{U}_{w_0+w_1}(1)}{\circ} \text{———} \underset{\hat{U}_{-2w_1}(1)}{\circ} \text{———} \underset{\hat{U}_{2w_1}(1)}{\circ}$$

It follows that the map $\mathrm{Sp}(V_{01})_{f_0} \to L_{01} \cap C(t_0)$ can be extended to an isomorphism $\mathrm{Sp}(V_{01}) \to L_{01}$ with $U_{-2w_0}(1) \to \tau$. Thus we have an extension of $\mathrm{Sp}(V)_{f_0} \to C(t_0)$ to $\mathrm{Sp}(V)_{f_0} \cup \mathrm{Sp}(V_{01}) \to C(t_0) \cup L_{01}$ such that $\mathrm{Sp}(V_{01}) \to L_{01}$ is an isomorphism. Similarly, we may construct an extension $\mathrm{Sp}(V)_{f_0} \cup \mathrm{Sp}(V_{02}) \to C(t_0) \cup L_{02}$ such that $\mathrm{Sp}(V_{02}) \to L_{02}$ is an isomorphism which maps $\mathrm{Sp}(V_0)$ to L_0. Replacing τ by τ^{b_0} if necessary, we may assume that $\mathrm{Sp}(V_{01}) \to L_{01}$ and $\mathrm{Sp}(V_{02}) \to L_{02}$ agree on $\mathrm{Sp}(V_{01}) \cap \mathrm{Sp}(V_{02}) = \mathrm{Sp}(V_0)$. This completes the argument.

Denote the map given in (4.5) by $x \to \hat{x}$.

(4.6) $\mathrm{Sp}(V)_{f_0} \cup \mathrm{Sp}(V_{01}) \cup \mathrm{Sp}(V_{02}) \to H$ may be extended to a map φ: $(\bigcup_{e \in V_0^\#} \mathrm{Sp}(V)_e) \cup \mathrm{Sp}(V_{01}) \cup \mathrm{Sp}(V_{02}) \to Y$ such that $\varphi|\mathrm{Sp}(V)_e$ is a homomorphism for all $e \in V_0^\#$.

PROOF. Set $u = U_{2w_0}(1)U_{-2w_0}(1)$ so that $\langle u \rangle = O_3(\mathrm{Sp}(V_0))$ acts regularly on $V_0^\#$. Thus if $e \in V_0^\#$, then for some i, $1 \leqslant i \leqslant 3$, $e = f_0 u^i$ and $\mathrm{Sp}(V)_e = \mathrm{Sp}(V)_{f_0}^i$. Hence, we can define a homomorphism $\mathrm{Sp}(V)_e \to Y$ by $x \to (\widehat{x^{u^{3-i}}})^{\hat{u}^i}$. It suffices to show that if $e \neq f$, then $\mathrm{Sp}(V)_e \to Y$ and $\mathrm{Sp}(V)_f \to Y$ agree on $\mathrm{Sp}(V)_e \cap \mathrm{Sp}(V)_f = \mathrm{Sp}(V_{123})$. But this follows immediately as u centralizes $\mathrm{Sp}(V_{123})$.

As before, set $\varphi(x) = \hat{x}$. Recall from §2 that a fundamental system for Φ is $\{p_0, p_1, p_2, p_3\}$ where $p_0 = 2w_0$ and $p_i = w_i - w_{i-1}$, $1 \leqslant i \leqslant 3$. Let $A_{ij} = \Phi \cap \langle p_i, p_j \rangle$ and set $A = \bigcup A_{ij}$, $0 \leqslant i, j \leqslant 3$. By (2.4), $U_\alpha(1)$ fixes a vector of $V_0^\#$ for each $\alpha \in \Phi$. In particular, $\{U_\alpha(1)\colon \alpha \in A\} \subseteq \bigcup_{e \in V_0^\#} \mathrm{Sp}(V)_e$. Therefore φ is defined on $\{U_\alpha(1)\colon \alpha \in A\}$ and we have the corresponding subset $\{\hat{U}_\alpha(1)\colon \alpha \in A\}$ of Y.

The next result is a direct application of Curtis' theorem (see (2.5)).

(4.7) φ can be extended to a homomorphism $\mathrm{Sp}(V) \to Y$ provided $(*)$ holds;

$(*)$ $[\hat{U}_\alpha(1), \hat{U}_\beta(1)] = \varphi([U_\alpha(1), U_\beta(1)])$ where α and β are linearly independent roots of Φ which are linear combinations of the same two fundamental roots.

Let $\{\alpha, \beta\}$ be a pair of independent roots in some A_{ij}. If $\langle U_\alpha(1), U_\beta(1)\rangle$ is contained in one of the groups $\mathrm{Sp}(V)_e$, $e \in V_0^\#$, or $\mathrm{Sp}(V_{0i})$, $i = 1, 2$, then evidently condition $(*)$ holds. In particular, $\langle U_\alpha(1), U_\beta(1)\rangle \subseteq \mathrm{Sp}(V)_e$ for some $e \in V_0^\#$ if $\{\alpha, \beta\} \subseteq A_{ij}$ with A_{ij} one of A_{02}, A_{03}, A_{13} or A_{23}. For example, suppose $\{\alpha, \beta\} \subseteq A_{02} = \{\pm p_0, \pm p_2\} = \{\pm 2w_0, \pm(w_2 - w_1)\}$. Without loss, assume that α involves p_0 and β involves p_2 so that $\alpha = \pm p_0$ and $\beta = \pm p_2$. But then $U_\alpha(1) = U_{\pm 2w_0}(1)$ fixes a vector $e \in V_0^\#$ whereas $U_\beta(1) = U_{\pm(w_2 - w_1)}(1)$ centralizes V_0 and so $\langle U_\alpha(1), U_\beta(1)\rangle \subseteq \mathrm{Sp}(V)_e$. A similar argument may be used for the cases A_{03}, A_{13} or A_{23}. On the other hand, $\mathrm{Sp}(V_{01}) = \langle U_\alpha(1)\colon \alpha \in A_{01}\rangle$. Hence, we restrict our attention to the case where

$$\begin{aligned}\{\alpha, \beta\} \subseteq A_{12} &= \{\pm p_1, \pm p_2, \pm(p_1 + p_2)\}\\ &= \{\pm(w_1 - w_0), \pm(w_2 - w_1), \pm(w_2 - w_0)\}.\end{aligned}$$

It is clear that $\langle U_\alpha(1), U_\beta(1)\rangle$ fixes some vector of $V_0^\#$ unless $\{\alpha, \beta\}$ is one of the pairs $\{-p_1, p_1 + p_2\} = \{w_0 - w_1, w_2 - w_0\}$ or $\{p_1, -p_1 - p_2\} = \{w_1 - w_0, w_0 - w_2\}$. In order to prove that $(*)$ holds in these cases as well, we require some additional analysis.

(4.8) $\hat{U}_{2w_1}(1)\hat{U}_{w_1 - w_2}(1) \not\sim_G \hat{U}_{w_1 - w_0}(1)$.

PROOF. Since $\hat{U}_{2w_1}(1)\hat{U}_{w_1 - w_2}(1) \in L_{12}' \cong A_6$ and $C(\langle b_0, b_3\rangle) = \langle b_0, b_3\rangle \times L_{12}$, it follows that $\langle b_0, b_3\rangle$ is self-centralizing in a Sylow 3-subgroup of $C(\hat{U}_{2w_1}(1)\hat{U}_{w_1 - w_2}(1))$. But $\{\langle b_0\rangle, \langle b_3\rangle\} = \langle b_0\rangle^G \cap \mathcal{E}_1(\langle b_0, b_3\rangle)$ by Proposition 3.1(i), hence $\langle b_0, b_3\rangle$ is a Sylow 3-subgroup of $C(\hat{U}_{2w_1}(1)\hat{U}_{w_1 - w_2}(1))$. On the other hand, $C(\hat{U}_{w_1 - w_2}(1)) \supseteq C_{L_{01}}(\hat{U}_{w_1 - w_2}(1)) \times \langle b_2, b_3\rangle \cong Z_2 \times \Sigma_4 \times E_{3^2}$. Hence $C(\hat{U}_{w_1 - w_2}(1))$ contains an E_{3^3}-subgroup and so $\hat{U}_{w_1 - w_2}(1) \not\sim_G \hat{U}_{w_1 - w_2}(1)$.

(4.9) $C_{L_{012}}(\hat{U}_{2w_1}) = \langle \hat{U}_{2w_1}, \hat{U}_{w_1 \pm w_0}, \hat{U}_{w_1 \pm w_2}\rangle L_{02}$.

PROOF. We first claim that L_{02} centralizes $\hat{U}_{2w_1}$. To prove this, observe that $L_{02} = \varphi(\mathrm{Sp}(V_{02})) = \varphi(\langle U_\alpha\colon \alpha \in \Phi \cap \langle w_0, w_2\rangle\rangle)$. If $\alpha \in \Phi \cap \langle w_0, w_2\rangle$, then $[U_\alpha, U_{2w_1}] = 1$. Also, $U_\alpha \subseteq \mathrm{Sp}(V)_e$ for some $e \in V_0^\#$ and U_{2w_1} centralizes V_0 implies that $\langle U_\alpha, U_{2w_1}\rangle \subseteq \mathrm{Sp}(V)_e$. Hence φ is defined on $\langle U_\alpha, U_{2w_1}\rangle$ and it follows that $\hat{U}_{2w_1}$ centralizes $\langle \hat{U}_\alpha\colon \alpha \in \Phi \cap \langle w_0, w_2\rangle\rangle = L_{02}$.

Let $\alpha \in A_{02}$. Then U_α and $U_{w_1 + w_2}$ centralize a common vector $e \in V_0^\#$ so that φ is defined on $\langle U_\alpha, U_{w_1 + w_2}\rangle$. Hence we have that $[\langle \hat{U}_{\pm 2w_0}\rangle, \hat{U}_{w_1 + w_2}] = 1 = [\hat{U}_{2w_2}, \hat{U}_{w_1 + w_2}]$. Also $[\hat{U}_{-2w_1}, \hat{U}_{w_1 + w_2}] \subseteq \hat{U}_{2w_1}\hat{U}_{w_1 - w_2}$. Thus $\langle \hat{U}_{\pm 2w_0}, \hat{U}_{\pm 2w_2}\rangle \cong \Sigma_3 \times \Sigma_3$ normalizes $\langle \hat{U}_{2w_1}, \hat{U}_{w_1 \pm w_2}\rangle$. Since $C_{L_{012}}(\hat{U}_{2w_1}) = O_2(C_{L_{012}}(\hat{U}_{2w_1}))L_{02}$ and L_{02} has 2-local 3 rank 1, it then follows that $\langle \hat{U}_{2w_1}, \hat{U}_{w_1 \pm w_2}\rangle \subseteq O_2(C_{L_{012}}(\hat{U}_{2w_1}))$. Similarly $\langle \hat{U}_{\pm 2w_0}, \hat{U}_{\pm 2w_2}\rangle$ normalizes $\langle \hat{U}_{2w_1}, \hat{U}_{w_1 + w_0}\rangle$ and so $\langle \hat{U}_{2w_1}, \hat{U}_{w_1 \pm w_0}\rangle \subseteq O_2(C_{L_{012}}(\hat{U}_{2w_1}))$ as well. Therefore $\langle \hat{U}_{2w_1}, \hat{U}_{w_1 \pm w_0}, \hat{U}_{w_1 \pm w_2}\rangle = O_2(C_{L_{012}}(\hat{U}_{2w_1}))$ proving the result.

The importance of (4.9) is that one of the missing relations $[\hat{U}_{w_1 - w_0}(1), \hat{U}_{w_0 - w_2}(1)]$ may be found within the subgroup $C_{L_{012}}(\hat{U}_{2w_1})$. Set $S = O_2(C_{L_{012}}(\hat{U}_{2w_1}))$. Now L_{02} contains the subgroup $\langle \hat{U}_{\pm(w_0 + w_2)}\rangle \times \langle \hat{U}_{\pm(w_0 - w_2)}\rangle \cong \Sigma_3 \times \Sigma_3$. As $O_3(\langle \hat{U}_{\pm(w_0 - w_2)}\rangle) \not\sim_{L_{02}} O_3(\langle \hat{U}_{\pm 2w_0}\rangle)$, it follows that $C_S(O_3(\langle \hat{U}_{\pm(w_0 - w_2)}\rangle)) = \hat{U}_{2w_1}$. This in turn implies that $C_S(\hat{U}_{\pm w_0 \pm w_2}) \cong E_8$. By checking to see that φ is defined on the appropriate subgroups, we have $C_S(\hat{U}_{-w_0 - w_2}) = \langle \hat{U}_{2w_1}, \hat{U}_{w_1 - w_0}, \hat{U}_{w_1 - w_2}\rangle$. Since $\langle \hat{U}_{\pm(w_0 - w_2)}\rangle$ centralizes $\hat{U}_{-w_0 - w_2}$, we may set $C_S(\hat{U}_{-w_0 - w_2}) = \hat{U}_{2w_1} \times S_0$ where

$S_0 = [C_S(\hat{U}_{-w_0-w_2}), \langle \hat{U}_{\pm(w_0-w_2)}\rangle]$. From $|C_{S_0}(\hat{U}_{-w_0+w_2})| = 2$, we deduce that S_0 is generated by an orbit of length 2 of $\hat{U}_{-w_0+w_2}$ acting on $C_S(\hat{U}_{-w_0-w_2})$. Hence $S_0 = \langle \hat{U}_{w_1-w_2}, \hat{U}_{w_1-w_0}\rangle$ or $\langle \hat{U}_{w_1-w_0}, \hat{U}_{2w_1}(1)\hat{U}_{w_1-w_2}(1)\rangle$. In the latter case, we have $\hat{U}_{w_1-w_0}(1) =_G \hat{U}_{2w_1-w_2}(1)$ against (4.7). Thus $S_0 = \langle \hat{U}_{w_1-w_2}, \hat{U}_{w_1-w_0}\rangle$. As $C_{S_0}(\hat{U}_{w_0-w_2}) = \hat{U}_{w_1-w_2}$, $\hat{U}_{w_0-w_2}(1)$ interchanges $\hat{U}_{w_1-w_0}(1)$ with $\hat{U}_{w_1-w_0}(1)\hat{U}_{w_1-w_2}(1)$ and we have $[\hat{U}_{w_0-w_2}(1), \hat{U}_{w_1-w_0}(1)] = \hat{U}_{w_1-w_2}(1)$. This shows that $(*)$ of (4.7) holds.

Finally, let γ be an element of the Weyl group of $\mathrm{Sp}(V)$ which interchanges w_1 with w_2 and centralizes $\langle w_0, w_3\rangle$. As $\gamma \in \mathrm{Sp}(V_{123})$, $\hat{\gamma}$ is defined and $\hat{U}_{w_2-w_1}(1) = [\hat{U}_{w_0-w_2}(1), \hat{U}_{w_1-w_0}(1)]^{\hat{\gamma}} = [\hat{U}_{w_0-w_2}(1)^{\hat{\gamma}}, \hat{U}_{w_1-w_0}(1)^{\hat{\gamma}}] = [\hat{U}_{w_0-w_1}(1), \hat{U}_{w_2-w_0}(1)]$ as required.

We have shown that $(*)$ of (4.7) holds in all cases. Hence φ may be extended to a homomorphism $\hat{\varphi}$: $\mathrm{Sp}(V) \to Y$. As $\hat{\varphi}(\mathrm{Sp}(V_{012})) = L_{012}$ and $Y = \langle C(t_0), L_{012}\rangle$, $\hat{\varphi}$ is onto. Therefore $\hat{\varphi}$ is an isomorphism by the simplicity of $\mathrm{Sp}(V)$. Noting that $\hat{\varphi}(\mathrm{Sp}(V_i)) = L_i$, $0 \leq i \leq 3$, we may summarize our results as follows.

(4.10) There exists an isomorphism $\hat{\varphi}$: $\mathrm{Sp}(V) \to Y$ such that $\hat{\varphi}(\mathrm{Sp}(V_A)) = L_A$, $A \in \mathcal{S}$.

In order to complete the proof of Theorem A, we must show that $Y = G$. The argument is the same as in [6] and employs Holt's theorem [2] with respect to the action of G on the set Ω of left cosets of Y. In order to apply Holt's theorem, we must verify that t_0 is central in a Sylow 2-subgroup of G. Let S be a Sylow 2-subgroup of $C(t_0)$. Then an easy calculation yields $Z(S) = \langle t_0, s\rangle$ where $t_0 \neq_Y s \neq_Y t_0 s \neq_Y t_0$. Thus if $t_0^G \cap Y = t_0^Y$ then not only does $S \in \mathrm{Syl}_2(G)$, but t_0 fixes precisely one point of Ω as well. Hence the hypotheses of Holt's theorem apply and we must then have $G = Y$. Before proving that $t_0^G \cap Y = t_0^Y$, we require the following elementary result.

(4.11) Let β be an element of order 3 of Y. Then $\beta^G \cap Y = \beta^Y$.

PROOF. Proposition 3.1(i) and (4.4) yield $N(B) = N_Y(B)$. Since B is the unique E_{3^4}-subgroup of $N(B)$ by [5, (2.7)], it follows that $N(B)$ contains a Sylow 3-subgroup P of G and $B = J(P)$. Hence $N(B)$ controls fusion in B. But every element of order 3 of Y is conjugate in Y to an element of B by (2.1i) and the result follows directly.

(4.12) $t_0^G \cap Y = t_0^Y$.

PROOF. Suppose there exists $\tau \in I(Y)$ such that $\tau =_G t_0$, but $\tau \neq_Y t_0$. By (2.1v), $C_Y(\tau)$ contains a Y-conjugate β of b_1 or $b_1 b_2$, so assume without loss that $\beta \in \{b_1, b_1 b_2\}$. Let $\tau^g = t_0$, $g \in G_0$. Then $\beta^g \in C(t_0)$, hence by (4.3), $\beta^g \in_{C(t_0)} L$. Since (4.11) implies that $\beta^g =_L \beta$, there exists $g_1 \in L$ such that $\beta^{gg_1} = \beta$. In particular, $\tau =_{C(\beta)} t_0$. But for $\beta = b_1$ or $b_1 b_2$, we see from (4.2) that $C(\beta) = O_{3'}(C(\beta))C_Y(\beta)$ where $O_{3'}(C(\beta))$ has odd order and this leads to a contradiction.

As noted earlier, we may now apply Holt's theorem to obtain $G = Y$ and the proof of Theorem A in this case is completed.

5. Proof of Theorem A: The $F_4(2)$ case. We assume henceforth that case (ii) of Proposition 3.1 holds. In this section we derive results (5.7), (5.8) and (5.9) which are used in the construction of $F_4(2)$ in §6. Recall from Proposition 3.1 that

$X = N(B)/C(B)$ contains the monomial group M with respect to the basis $\{b_0, b_1, b_2, b_3\}$.

(5.1) $N(B)$ controls fusion in B.

The fusion patterns corresponding to cases (iia) and (iib) of Proposition 3.1 are as follows:

(a) $b_0^X = b_0^M \cup (b_0b_1b_2b_3)^M$, $(b_0b_1)^X = (b_0b_1)^M$, $(b_1b_2b_3)^X = (b_1b_2b_3)^M$;

(b) $b_0^X = b_0^M \cup (b_0b_1)^M \cup (b_0b_1b_2b_3)^M$, $(b_1b_2b_3)^X = (b_1b_2b_3)^M$.

Furthermore the group X occurring in case (a) is a subgroup of the group X occurring in case (b).

PROOF. We know that X is an extension of $Q_8 * Q_8$ by $\Sigma_3 \times \Sigma_3$ or $\Sigma_3 \wr Z_2$ in cases (a) or (b) respectively. Since the orbits of X acting on B are unions of the orbits of M, it follows easily that the orbits of X are as described.

In order to prove that $N(B)$ controls fusion in B, it suffices to show that if $P \in \mathrm{Syl}_3(N(B))$, then $B = J(P)$. Suppose in fact that $B \neq B^* \subseteq P$ with $B^* \cong E_34$. It is clear that B is the unique E_34-subgroup of $P \cap C(b_0)$, so we have $B^* \not\subseteq C(b_0)$. Thus $b_0 \notin B^*$. In fact, $b_0^{N(B)} \cap B^* = \varnothing$. Assume that $[B\colon B^* \cap B] = 3$. Then $B^* \cap \langle b_0, b_1b_2b_3\rangle \neq 1$, so $b_1b_2b_3 \in B^*$. A similar argument shows that $\langle b_0b_2b_3, b_1^2b_2b_3\rangle \subseteq B^*$. Thus $b_0 = (b_1b_2b_3)(b_0b_2b_3)(b_1^2b_2b_3) \in B^*$, a contradiction. Since $|B^* \cap B| \leqslant 9$, B^* covers P/B. Hence we may assume, without loss, that B^* contains an element β which cyclically permutes $\{b_1, b_2, b_3\}$ and centralizes b_0. Then $|C_B(\beta)| = 9$ and $b_0 \in C_B(\beta)$, so $B^* \cap B \subset C_B(\beta)$ and $|B^* \cap B| = 3$, a contradiction. This shows that $B = J(P)$ as required.

The last statement follows from remarks made in [**5**, §5].

Both fusion patterns have the convenient property that B contains a subgroup $A_1 = \langle b_0, b_1b_2b_3\rangle \cong E_{3^2}$ such that every E_9-subgroup of A_1 contains an element of b_0^X. Taking $s_0 \in \mathrm{Inv}(C(A_1))$, the usual generational arguments and the assumption that $O_2(C(s_0)) = F^*(C(s_0))$ then allow us to construct $O_2(C(s_0))$ in (5.5) and (5.6). From this, we find the chief factors of $C(s_0)$ and show that $C(b_2b_3^2) \cong C(b_0)$ in (5.7) (of course in case (b), we already have $C(b_2b_3^2) =_G C(b_0)$). In (5.8), we use (5.7) to determine $O_2(C(t_0))$ and find a chief series for $C(t_0)$. (Here, as before, t_0 inverts b_0 and centralizes L.) Finally, in (5.9), we produce an $SL(3, 2)$-subgroup R of L with $b_2b_3^2 \in R$ such that $C(R) \cong SL(3, 2)$.

We shall use the following notation in this section. $A_1 = \langle b_0, b_1, b_2b_3\rangle$ and s_0 is a fixed involution in $C(A_1)$ which inverts $b_2b_3^2$. Note that s_0 is a short root involution of L regarded as $C_3(2)$.

In order to analyze $C(s_0)$, we need to obtain information about $N(A_1)$.

(5.2) $N(A_1) = \langle b_2b_3^2, s_0\rangle \times H$ where $H = C(\langle b_2b_3^2, s_0\rangle) \cap N(A_1)$. $\mathrm{Aut}_H(A_1)$ acts as the monomial group (on A_1) with respect to the basis $\{b_0b_1, b_0b_1^2, b_2b_3\}$. In particular, every hyperplane of A_1 contains an H-conjugate of $\langle b_0\rangle$.

PROOF. It follows from (3.2) that $C(A_1) = O_{3'}(C) \times A_1 \times \langle b_2b_3^2, s_0\rangle$. Thus $\langle b_2b_3^2, s_0\rangle = O^{2'}(C(A_1))$ is normal in $N(A_1)$. An easy argument shows that $N(A_1) = \langle b_2b_3^2, s_0\rangle \times H$ with H as described. Since $\mathrm{Aut}(B)$ contains a unique involution, namely the image of s_0, that inverts $b_2b_3^2$ and centralizes a hyperplane of B, we now have that $N(A_1) = N(\langle b_2b_3^2\rangle) \cap N(B)$.

If fusion pattern (b) occurs, then by the last statement in (5.1) we can choose $\tau \in N(B)$ such that $b_0^\tau = b_2b_3^2$ and τ interchanges $\langle b_0\rangle^M \cup \langle b_0b_1b_2b_3\rangle^M$ with $\langle b_0b_1\rangle^M$. Then $B_1^\tau = A_1$ by the previous paragraph and, since $\{\langle b_1\rangle, \langle b_2\rangle, \langle b_3\rangle\}^\tau = \langle b_0b_1\rangle^M \cap \mathfrak{S}_1(A_1) = \{\langle b_0b_1\rangle, \langle b_0b_1^2\rangle, \langle b_2b_3\rangle\}$, the action of $\mathrm{Aut}_H(A_1)$ on A_1 must be as described.

Assume now that fusion pattern (a) occurs. Then $\langle b_0b_1\rangle^G \cap \mathfrak{S}_1(A_1) = \{\langle b_0b_1\rangle, \langle b_0b_1^2\rangle, \langle b_2b_3\rangle\}$ and $N(A_1)$ must act on this set. Using the monomial action of $N(B)$ on the basis $\{b_0, b_1, b_2, b_3\}$, it is easy to find a subgroup $K = \langle b_0b_1, s_1\rangle \times \langle b_2b_3, s_2\rangle \times \langle b_0b_1^2, s_3\rangle$ of $N(A_1) \cap C(\langle b_2b_3^2, s_0\rangle)$ with $K \cong \Sigma_3 \times \Sigma_3 \times \Sigma_3$. As $N(B) \cap N(\langle b_2b_3^2\rangle) = N(A_1)$ has index 12 in $N(B)$ and $|\mathrm{Aut}(B)| = 2^7 \cdot 3^2$, we have that $K \cdot C(A_1)$ has index 6 in $N(A_1)$. It follows that $\mathrm{Aut}_H(A_1)$ acts monomially on the basis $\{b_1b_1, b_0b_1^2, b_2b_3\}$.

The last statement follows easily from the observation that $b_0^2 = (b_0b_1) \cdot b_0b_1^2$ and inspection of the E_9-subgroups of a group $\langle a_1, a_2, a_3\rangle \cong E_{27}$.

We now derive two short results before attacking the structure of $O_2(C(s_0))$.

(5.3) $O_{3'}(C(b_0)) = 1$. Thus $C(b_0) = \langle b_0\rangle \times L$ and $H \cong \Sigma_3 \wr \Sigma_3$.

PROOF. Set $Z = O_{3'}(C(b_0))$. Then Z has odd order by (3.2i). Since $O_{3'}(H) = 1$ for every 3-local subgroup H of L, it follows that $Z = O_{3'}(C(B))$ and that $Z = O_{3'}(C(\Gamma))$ for every $\Gamma < B$ that contains a conjugate of b_0. In particular, $Z = O_{3'}(C(\Gamma))$ for all hyperplanes Γ of A_1. As $Z \subseteq C(s_0)$, this implies that Z centralizes $\langle C(\Gamma) \cap O_2(C(s_0))\colon \Gamma \in \mathfrak{S}_2(A_1)\rangle = O_2((C(s_0)))$. Thus $Z = 1$ because G has characteristic 2-type. We now have that $C(A_1) = A_1\langle b_2b_3^2, s_0\rangle$, so that H/A_1 acts faithfully on A_1. It follows that $H \cong \Sigma_3 \wr \Sigma_3$.

(5.4) Set $J = O^3(C(\langle b_2, b_3\rangle))$. Then $J \cong \mathrm{Sp}(4, 2)$, J centralizes s_0 and $\langle b_0, b_1\rangle \subseteq J$.

PROOF. By (5.3), we have $C(\langle b_0, b_1\rangle) = \langle b_0\rangle \times C_L(b_1) = \langle b_0, b_1\rangle \times J^*$, where $J^* = O^3(C(\langle b_0, b_1\rangle)) \cong \mathrm{Sp}(4, 2)$ and $\langle b_2, b_3\rangle \subseteq J^*$. Since $M \subseteq \mathrm{Aut}_G(B)$, there exists $\alpha \in N(B)$ such that α interchanges $\langle b_0, b_1\rangle$ with $\langle b_2, b_3\rangle$. Setting $J = (J^*)^\alpha$, we have that $\langle b_0, b_1\rangle \subseteq J$ and $J = O^3(C(\langle b_2, b_3\rangle)) \cong \mathrm{Sp}(4, 2)$. As $s_0 \in N(\langle b_2, b_3\rangle)$, s_0 acts on J. But s_0 centralizes $\langle b_0, b_1\rangle$ and from the structure of $\mathrm{Aut}(\mathrm{Sp}(4, 2)) \cong P\Gamma L(2, 9)$, it follows that s_0 centralizes J.

(5.5) Let $S = O_2(C(s_0))$. Then $S = S_0 \times S_1$ where $S_0 \cong E_{2^6}$ and $S_1 \cong X_{2^9}^+$.

PROOF. Let $\Gamma_0 = \langle b_0, b_1\rangle$, $\Gamma_1 = \langle b_0, b_1b_2b_3\rangle$, and $\Gamma_2 = \langle b_0, b_2b_3\rangle$. Then $\{\Gamma_i\colon i = 0, 1, 2\}$ is a set of representatives of H-orbits on $\mathfrak{S}_2(A_1)$. We can compute the centralizer of each Γ_i as follows:

$$C(\Gamma_0) = \langle b_0\rangle \times C_L(b_1) \cong E_9 \times \mathrm{Sp}(4, 2),$$
$$C(\Gamma_1) = \langle b_0\rangle \times C_L(b_1b_2b_3) \cong Z_3 \times GU(3, 2),$$
$$C(\Gamma_2) = \langle b_0\rangle \times C_L(b_1b_2) \cong E_9 \times \Sigma_3 \times \Sigma_3.$$

Since $\langle b_1, b_2b_3\rangle < C_L(s_0)$, we can compute $T_i := O_2(C(s_0) \cap C(\Gamma_i))$, $i = 0, 1, 2$. We have $T_0 \cong E_8$, while $T_1 \cong Q_8$ and $T_2 = \langle s_0\rangle$. It is clear that $C_S(\Gamma_i) < T_i$, $i = 0, 1, 2$. Thus $C_S(A_1) = \langle s_0\rangle$ and H acts faithfully on all nontrivial H-sections of $S/\langle s_0\rangle$. By (3.5), the irreducible H-sections of $S/\langle s_0\rangle$ have orders either 2^6 or 2^8 and there is at most one section of each order. We argue that H does not act irreducibly on $S/\langle s_0\rangle$.

Suppose first that $S/\langle s_0\rangle$ has order 2^6. Then $C(s_0)/S$ embeds into $GL(6, 2)$. Setting $\bar{S} = S/\langle s_0\rangle$, we have $C_{\bar{S}}(b_0^{\pm 1}b_1b_2b_3) \cong C_{\bar{S}}(b_0) \cong E_4$ and $C_{\bar{S}}(b_1b_2b_3) = 1$ by (5.2) and (3.5). But $C_{C(s_0)/S}(\langle b_0, b_1b_2b_3\rangle)$ contains an A_4-subgroup. On the other hand, every E_9-subgroup of $GL(6, 2)$ whose centralizer contains an A_4-subgroup must contain an element with 4-dimensional fixed point subspace in the natural representation, a contradiction.

Now suppose that $S/\langle s_0\rangle$ has order 2^8. Then $T_1 \leqslant S$ and S is nonabelian. The irreducible action of H on $S/\langle s_0\rangle$ forces S to be extra-special of type $+$. We can thus embed $C(s_0)/S$ into $O^+(8, 2)$. By (5.2) and (3.5) we have $C_{\bar{S}}(b_0) \cong C_{\bar{S}}(b_1) \cong X_{2^5}^+$ and $C_S(b_0b_1) = C_S(b_0b_1^2) = \langle s_0\rangle$. By the structure of $O^+(8, 2)$, it follows that $C_{C(s_0)/S}(\langle b_0, b_1\rangle)$ is a 3-group. But $C_{C(s_0)/S}(\langle b_0, b_1\rangle)$ contains an A_4-subgroup of $(C(s_0) \cap C(\langle b_0, b\rangle))/\langle s_0\rangle$, a contradiction.

Thus S has order 2^{15} and contains an H-invariant subgroup $S^* < S$ such that H acts irreducibly on $S^*/\langle s_0\rangle$. By (3.5), either $|S^*/\langle s_0\rangle| = 2^6$ or $|S^*/\langle s_0\rangle| = 2^8$. In the former case, $S^* = \langle C_S(\Gamma_0)^H\rangle$ and $S = S^*\langle C_S(\Gamma_1)^H\rangle$ while in the latter case, $S^* = \langle C_S(\Gamma_1)^H\rangle$ and $S = S^*\langle C_S(\Gamma_0)^H\rangle$. Since S is nilpotent, we have $[S, S^*] \leqslant \langle s_0\rangle$. Letting Γ be H-conjugate to Γ_0 and Δ be H-conjugate to Γ_1, either $C_S(\Gamma) \leqslant S^*$ or $C_S(\Delta) \leqslant S^*$. In either case $[C_S(\Delta), C_S(\Gamma)] \leqslant \langle s_0\rangle \leqslant C_S(\Gamma)$. Thus $C_S(\Delta)$ centralizes $C_S(\Gamma)$ by (3.4). It follows that $\langle C_S(\Gamma_0)^H\rangle$ centralizes $\langle C_S(\Gamma_1)^H\rangle$ and so $S = S^* \cdot C_S(S^*)$. If $|S^*| = 2^7$, then $S^* = \langle C_S(\Gamma_0)^H\rangle$ is abelian by (3.4), and $S^* \leqslant Z(S)$. In fact $S^* = Z(S)$ because $C_S(\Gamma_1) \leqslant S$ is nonabelian. If $|S^*| = 2^9$, then $C_S(\Gamma_1) \leqslant S^*$ is nonabelian, so $\langle s_0\rangle = Z(S^*)$ and $C_S(S^*)$ has order 2^7. As $C_S(S^*)$ is clearly H-invariant, we can replace S^* by $C_S(S^*)$ in this case and conclude that $Z(S)$ has order 2^7 in any case.

This also shows that $S/\langle s_0\rangle$ is abelian, so every subgroup of S containing s_0 is normal in S. In particular, $Q_8 \cong C_S(\Gamma_1) \trianglelefteq S$. Hence, $S_1 = \langle C_S(\Gamma_1)^H\rangle$ is normal in S and $S_1 \cong X_{2^9}^+$. Setting $S_0 = [Z(S), A_1]$, we have $S_0 \cong E_{2^6}$, $s_0 \notin S_0$ and $S_0 \trianglelefteq S$. Thus $S_0 \cap S_1 = 1$ and $S = S_0 \times S_1$.

Before tackling the structure of $C(s_0)$, we need another technical lemma.

(5.6) $C(s_0) \cap C(b_2b_3) = C_S(b_2b_3) \cdot J \cdot \langle b_2b_3\rangle$ where $C_S(b_2b_3) \cong E_{2^5}$ and $J = O^3(C(\langle b_2, b_3\rangle)) \cong \mathrm{Sp}(4, 2)$ (as in (5.4)). Furthermore, $C(Z(S)) = S$.

PROOF. By (5.2) and (3.5), b_2b_3 acts fixed point free on $S/Z(S)$ and $C_{Z(S)}(b_2b_3) \cong E_{2^5}$.

Set $D = C(s_0) \cap C(b_2b_3)$ and $\bar{D} = D/D \cap S$. As $O_{3'}(C(s_0) \cap C(\Gamma_i))$ is a 2-group for $i = 0, 1, 2$, we must have $S = O_{3'}(C(s_0))$. Also, $A_1 \in \mathrm{Syl}_3(C(b_0) \cap C(s_0))$ and $A_1 \cap C(Z(S)) = 1$ together imply that $|C(b_0) \cap C(Z(S))|_3 = 1$. Thus $C(Z(S)) \subseteq O_{3'}(C(s_0)) = S$ which proves that $C(Z(S)) = S$. As an immediate consequence, we have that $\bar{D}$ acts faithfully on both $Z(S)$ and $Z(S)/\langle s_0\rangle \cong E_{2^6}$. Since b_2b_3 centralizes an E_{16}-subgroup of $Z(S)/\langle s_0\rangle$, we may embed $\bar{D}$ in $C_{\mathrm{Aut}(S/\langle s_0\rangle)}(b_2b_3) \cong Z_3 \times GL(4, 2)$. We know from (5.4) that $\langle b_2b_3\rangle \times J \subseteq D$, hence $\bar{D}$ contains a $Z_3 \times \mathrm{Sp}(4, 2)$-subgroup. As $\mathrm{Sp}(4, 2)$ is a maximal subgroup of $GL(4, 2)$, we may verify that $\langle b_2b_3\rangle \times J$ covers $\bar{D}$ by eliminating the possibly $\bar{D} \cong Z_3 \times GL(4, 2)$. In this case, $\bar{D}$ contains a subgroup $\bar{\Gamma} \times \bar{K}$ with $\bar{\Gamma} \cong E_9$ and $\bar{K} \cong SL(2, 4)$. Because $A_1 \in \mathrm{Syl}_3(C(s_0) \cap C(b_2b_3))$, we may assume, without loss, that A_1 contains an E_9-subgroup Γ which maps onto $\bar{\Gamma}$. Then Γ contains a $C(s_0) \cap N(A_1)$ conjugate of

b_0 which forces $C(s_0) \cap C(b_0)$ to have order divisible by 5. But $C(b_0) = \langle b_0 \rangle \times L$ by (5.3) and s_0 is a short root involution of $L \cong C_3(2)$ yields $|C(s_0) \cap C(b_0)| = 3|C_L(s_0)| = 2^9 3^3$. This leads to a contradiction and the argument is completed.

We are now ready to prove the major result of this section.

(5.7) $C(\langle b_2 b_3^2, s_0 \rangle)$ contains a subgroup M isomorphic to Sp(6, 2) such that

(a) $C(s_0) = SM$ and

(b) $C(b_2 b_3^2) = \langle b_2 b_3^2 \rangle \times M$.

PROOF. Recall from (5.2) that $N(A_1) = \langle b_2 b_3^2, s_0 \rangle \times H$ with $A_1 \subseteq H$. Let $M = \langle J, H \rangle$. We will show that M has the desired properties. Setting $Z = Z(S)$ and $\bar{Z} = Z/\langle s_0 \rangle$, it follows that M acts on $\bar{Z}$ with $C_M(\bar{Z}) = M \cap S \subseteq O_2(M)$ by (5.6). But $B \subseteq N(M)$ and $m_{2,3}(G) = 3$ by (3.2). Therefore $O_2(M) = 1$ and M acts faithfully on $\bar{Z}$. In fact, M acts irreducibly on $\bar{Z}$ since H does.

Let $Z_1 = [Z, b_2 b_3]$ and $Z_2 = C_Z(b_2 b_3)$. Then $J \subseteq C(b_2 b_3)$ implies that J acts on Z_1 and Z_2. By (5.6), $\bar{Z}_1 \cong E_4$ and $\bar{Z}_2 \cong E_{16}$. As $\langle b_2 b_3 \rangle$ acts irreducibly on $\bar{Z}_1$, J centralizes $\bar{Z}_1$. Furthermore, J is generated by a class Ω of involutions which act as transvections on $\bar{Z}_2$. Thus the elements of Ω act as transvections on $\bar{Z}$. Let $M^* = \langle \Omega^M \rangle$ so that $M^* \trianglelefteq M$ and $J \subseteq M^*$. Since J acts irreducibly on $\bar{Z}_2$, a dimension argument shows that $\bar{Z}$ has precisely one M^*-invariant subspace of dimension at least 4. But this subspace must then be invariant under M, hence M^* acts irreducibly on $\bar{Z}$. By a theorem of McLaughlin **[12]**, M^* is isomorphic to $GL(6, 2)$, Sp(6, 2) or $O^{\pm}(6, 2)$. Since all these subgroups of Aut($\bar{Z}$) are self-normalizing, we have $M^* = M$. Also $C_M(b_2 b_3) = \langle b_2 b_3 \rangle \times J$ by (5.6). It then follows from inspection that $M \cong$ Sp(6, 2).

We now show that M covers $\widetilde{C(s_0)} = C(s_0)/S$. As $C_{C(s_0)}(\bar{Z}) = S$ by (5.6), it follows that $\widetilde{C(s_0)} = \mathrm{Aut}_{C(s_0)}(\bar{Z})$. An elementary consequence of McLaughlin's theorem **[12]** is that $\tilde{M}$ is maximal in Aut($\bar{Z}$). Thus it suffices to eliminate the case where $\widetilde{C(s_0)} = \mathrm{Aut}(\bar{Z})$. But this may be done, as in the previous paragraph, by observing that $C_{\widetilde{C(s_0)}}(\widetilde{b_2 b_3}) = G_{\tilde{M}}(\widetilde{b_2 b_3}) \cong Z_3 \times \mathrm{Sp}(4, 2)$ is incompatible with $C(s_0) = \mathrm{Aut}(\bar{Z})$.

Finally, we must prove that $C(b_2 b_3^2) = \langle b_2 b_3^2 \rangle \times M$. Set $C_1 = C(b_2 b_3^2)$ and $C_1^* = \langle b_2 b_3^2 \rangle \times M$. If $b_0 =_G b_2 b_3^2$, then $C(b_0) \cong C_1$, hence $C_1 = C_1^*$. Thus we may assume that $b_0 \neq_G b_2 b_3^2$ in which case fusion pattern (a) occurs. Let t_3 be an involution of $N(B)$ which inverts b_3 and centralizes $\langle b_0, b_1, b_2 \rangle$ and let $s^* = s_0^{t_3}$. Conjugating $C(s_0) \cap C(b_2 b_3) = \langle b_2 b_3 \rangle \times C_S(b_2 b_3) J$ by t_3 yields $C(s^*) \cap C(b_2 b_3^2) \cong Z_3 \times E_{2^5}\mathrm{Sp}(4, 2)$. Moreover, s^* inverts $b_2 b_3$ and centralizes $A_1^{t_3} = \langle b_0, b_1, b_2 b_3^2 \rangle$ implies that $s^* \in C(b_2 b_3^2) \cap N(A_1) = \langle b_2 b_3^2 \rangle \times H$ by (5.2). Hence $s^* \in H \subseteq M$ and we conclude that s^* is a transvection of M with $C_{C_1}(s^*) = C_{C_1^*}(s^*)$.

We claim that $(s^*)^{C_1} \cap C_1^* = (s^*)^{C_1^*}$. To prove this, we first observe that $C_{C_1}(b_0 b_1) = C_{C_1^*}(b_0 b_1)$. In fact, $b_0 b_1 =_M b_2 b_3$ and $C_{C_1}(b_2 b_3) = C_{C_1^*}(b_0)$. Now every involution of M centralizes a conjugate of b_0 or of $b_0 b_1$ by (2.1iv) (in the case at hand, b_0 and $b_0 b_1$ have their roles reversed). Let τ be an involution of C_1^* so that $\tau \in M$ and assume that $\tau^x = s^*$ for some $x \in C_1$. Without loss, we may assume that τ centralizes β with $\beta \in \{b_0, b_0 b_1\}$. Since β is inverted in $C_M(\tau)$ (see **[1]**), $\beta \in O^{2'}(C_M(\tau))$, hence $\beta^x \in O^{2'}(C_{C_1}(s^*)) = C_M(s^*)$. Furthermore, $\langle b_0, b_1 \rangle \in \mathrm{Syl}_3(C_M(s^*))$, thus we may replace β^x by a suitable $C_M(s^*)$ conjugate if necessary to insure that $\beta^x \in \langle b_0, b_1\rangle$. In fact, as case (a) holds and $\langle b_0, b_1\rangle$ is contained in a Sp(4, 2)-subgroup of $C_M(s^*)$, we may choose $\beta^x = \beta$. But then $x \in C_{C_1}(\beta) = C_{C_1^*}(\beta)$ and so $\tau =_{C_1^*} s^*$ as claimed.

Now let $S^* \in \mathrm{Syl}_2(C_{C_1}(s^*))$. Then $S^* \in \mathrm{Syl}_2(C_1^*)$, $Z(S^*) \cong E_4$ and $s^{*C_1} \cap Z(S^*) = \{s^*\}$ by the claim and an easy calculation. In particular, we have that $S^* \in \mathrm{Syl}_2(C_1)$. Setting $\overline{C_1} = C_1/O_{3'}(C_1)\langle b_2b_3\rangle$, we may apply Holt's theorem **[11]** to conclude that $\overline{M} = \overline{C}_1$. Thus $C_1 = O_{3'}(C_1)(\langle b_2b_3^2\rangle \times M)$. It remains for us to prove that $O_{3'}(C_1) = 1$. But every hyperplane of A_1 that contains $\langle b_0\rangle$ acts fixed point free on $O_{3'}(C_1)$ by (5.3). Since $A_1 < H < M$, the last statement of (5.2) implies that $O_{3'}(C_1) = 1$. This completes the proof of part (b).

We now analyze the structure of $C(t_0)$.

(5.8) $C(t_0) = TL$ where $T = O_2(C(t_0)) = T_0 \times T_1$, $T_0 \cong E_{2^6}$ and $T_1 \cong X_{2^9}^+$.

PROOF. First recall that $N(\langle b_0\rangle) = \langle b_0, t_0\rangle \times L$. This implies that $\langle b_0, t_0\rangle$ centralizes $b_2b_3^2$ and normalizes A_1, hence $\langle b_0, t_0\rangle \subseteq M$ by (5.2). Also, $C_M(\langle b_0, t_0\rangle) \cong \Sigma_3 \times \Sigma_3$ so that $\langle b_0, t_0\rangle$ is a short root $SL(2, 2)$-subgroup of M considered as $C_3(2)$. Beginning in (5.2) and culminating in (5.7), we analyzed the structure of $C(b_2b_3^2)$ and $C(s_0)$ assuming only that $(b_0, C(b_0), L)$ satisfies the hypotheses of Theorem A, that $\mathrm{Aut}_X(B)$ is as described in Proposition 3.1(ii) and that $\langle b_2b_3^2, s_0\rangle$ is a short root $SL_2(2)$-subgroup of L. By (5.7b), $(b_2b_3^2, C(b_2b_3^2), M)$ satisfies the hypotheses of Theorem A with B having the same role as before. Since $\langle b_0, t_0\rangle$ is a short root $SL(2, 2)$-subgroup of M, we may then argue by symmetry to conclude that the analogue of (5.7a) holds for the structure of $C(t_0)$. Thus $C(t_0) = TL$ with T as described by (5.5).

The final result of this section is crucial for our construction of $F_4(2)$.

(5.9) Let R be an $SL(3, 2)$-subgroup of L which contains $\langle b_2b_3^2, s_0\rangle$. Then $C(R) \cong SL(3, 2)$.

PROOF. As $\langle b_2b_3^2, s_0\rangle$ is a short root $SL(2, 2)$-subgroup of L regarded as $C_3(2)$, there do exist $SL(3, 2)$-subgroups which contain $\langle b_2b_3^2, s_0\rangle$. Suppose that R is one such subgroup. Let $\langle\gamma\rangle$ be a Sylow 7-subgroup of R normalized by $\langle b_2b_3^2\rangle$. By properties of Sp(6, 2), $C_L(\gamma) = \langle\gamma\rangle$, so $C_L(R) = 1$ and $C(R) \cap N(\langle b_0\rangle) = \langle b_0, t_0\rangle$. We consider the action of γ on $T = O_2(C(t_0))$ and set $V = C_T(\gamma)$. Since $Z(T) \cong E_{2^7}$ by (5.8) and L acts naturally on $Z(T)/\langle t_0\rangle$ by (2.2), $V \cap Z(T) = \langle t_0\rangle$. By (5.8) and Maschke's theorem, $T/\langle t_0\rangle = Z(T)/\langle t_0\rangle \times E/\langle t_0\rangle$ where E is $\langle\gamma\rangle$-invariant. Evidently $T = Z(T) * E$ with $Z(T) \cap E = \langle t_0\rangle$, hence $E \cong X_{2^9}^+$. It then follows from (3.4) that $V = C_E(\gamma)$ is extra-special and since γ acts faithfully on E, we have $V \cong D_8$.

Now $b_2b_3^2$ must centralize V, so $V \subseteq O^3(C(b_2b_3^2)) \subseteq C(s_0)$ by (5.7). This yields $R = \langle\gamma, b_2b_3^2, s_0\rangle \subseteq C(V)$. Thus $t_0 \in C(R)'$. As t_0 inverts $\langle b_0\rangle \subseteq C(R)$, it follows easily from a result of Feit-Thompson **[3]** that $C(R) \cong SL(3, 2)$.

6. Proof of Theorem A: The $F_4(2)$ case continued. In this section, we continue with our analysis of case (ii) of Proposition 3.1. Our first goal is to use the results of §5 to construct a subgroup of G isomorphic to $F_4(2)$.

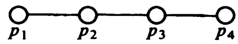
(6.1) G contains a subgroup $Y \cong F_4(2)$ with the following properties:

(i) $Y = \langle C(b_0), C(b_2b_3^2)\rangle$,

(ii) $Y = \langle C(t_0), C(s_0)\rangle$,

(iii) s_0 and t_0 are root involutions of Y of different lengths.

PROOF. We shall construct Y by applying (2.9). In §2 we constructed a root system Δ of type $F_4(2)$ with fundamental system $\{p_1, p_2, p_3, p_4\}$ as indicated by the following diagram:



Associated with the fundamental roots are root $SL(2, 2)$-subgroups $\langle U_{\pm p_i}\rangle$, $1 \leq i \leq 4$. Here $\langle U_{\pm p_i}\rangle$ is a short root $SL(2, 2)$-subgroup, $i = 1, 2$, whereas $\langle U_{\pm p_i}\rangle$ is a long root $SL(2, 2)$-subgroup for $i = 3, 4$.

Set $K_0 = \langle b_0, t_0\rangle$, $K_2 = \langle b_1, t_1\rangle$ and $K_4 = \langle b_2b_3^2, s_0\rangle$. Recall that t_1 is an involution of $N_L(B)$ which inverts b_1 and centralizes $\langle b_0, b_2, b_3\rangle$. We now identify L with $B_3(2)$ so that K_2 is a short root $SL(2, 2)$-subgroup and K_4 is a long root $SL(2, 2)$-subgroup. Let π_2: $L \to F_4(2)$ be a monomorphism with $\pi_2(L) = \langle U_{\pm p_j}: 2 \leq j \leq 4\rangle$. We may choose π_2 such that $\pi_2(K_4) = \langle U_{\pm p_4}\rangle$. Since K_2 is the unique short root $SL(2, 2)$-subgroup of L which centralizes K_4 by (2.8), it follows that $\pi_2(K_2) = \langle U_{\pm p_2}\rangle$. Set $K_3 = \pi_2^{-1}(\langle U_{\pm p_3}\rangle)$. Then $\langle K_3, K_4\rangle$ is an $SL(3, 2)$-subgroup of L which contains $K_4 = \langle b_2b_3^2, s_0\rangle$, hence by (5.9), $C(\langle K_3K_4\rangle) \cong SL(3, 2)$. If we regard M as $C_3(2)$, we then have that $C(\langle K_3, K_4\rangle)$ is an $SL(3, 2)$-subgroup of M which contains the short root $SL(2, 2)$-subgroup K_0.

We know from (2.7ii) that all $SL(3, 2)$-subgroups of M which contain a root $SL(2, 2)$-subgroup are conjugate. Furthermore, as was shown in (2.7ii), M contains an $A_3(2)$-subgroup whose root $SL(2, 2)$-subgroups are short root $SL(2, 2)$-subgroups of M. Thus there exist short root $SL(2, 2)$-subgroups K_1 and K_2^* of M such that $\langle K_0, K_1, K_2^*\rangle \cong A_3(2)$, $\langle K_0, K_1\rangle = C(\langle K_3, K_4\rangle)$ and $\{K_0, K_1, K_2^*\}$ is an $A_2(2)$ generating system of type A_3 for $\langle K_0, K_1, K_2^*\rangle$ (see [5, §6]). In particular, this implies that $[K_0, K_2^*] = 1$ and also that there exists an isomorphism π_1: $\langle K_1, K_2^*\rangle \to \langle U_{\pm p_1}, U_{\pm p_2}\rangle$ with $\pi_1(K_1) = \langle U_{\pm p_1}\rangle$ and $\pi_1(K_2^*) = \langle U_{\pm p_2}\rangle$. Since K_2^* is a short root $SL(2, 2)$-subgroup of M which commutes with K_0, it follows from (2.8) that $K_2 = K_2^*$. Now set $Y = \langle K_1, K_2, K_3, K_4\rangle$. The previous discussion shows that Y satisfies the hypotheses of (2.9). Thus there exists an epimorphism π: $Y \to F_4(2)$ with $\pi(K_i) = \langle U_{\pm p_i}\rangle$, $1 \leq i \leq 4$, and $\ker(\pi) \subseteq Z(Y)$. But $C_Y(b_2b_3^2) = \langle b_2b_3^2\rangle \times M = C(b_2b_2^2)$ by (5.7), hence $\ker(\pi) = 1$ and π is an isomorphism.

Evidently, π identifies t_0 and s_0 as being root involutions of different lengths. Applying (5.7) and (5.9) in conjunction with (2.10) yields $C_Y(t_0) = C(t_0)$ and $C_Y(s_0) = C(s_0)$. Since these groups are distinct maximal parabolic subgroups, we have $Y = \langle C(t_0), C(s_0)\rangle$. Finally, $L \subseteq C(t_0)$, hence $C(b_0) = \langle b_0\rangle \times L \subseteq Y$. An easy argument then gives $Y = \langle C(b_0), C(b_2b_3^2)\rangle$ and so all parts are now proved.

(6.2) Fusion pattern (b) of Proposition 3.1(ii) does not occur. Also, $N(B) \subseteq Y$.

PROOF. Assume the contrary. We claim that $N(B) \subseteq N(Y)$. First note that $B \subseteq Y$ by (6.1i). Furthermore, $\mathrm{Aut}_Y(B) = N_Y(B)/B$ has order at least $2^7 \cdot 3^2$. This follows from the existence of a subgroup of $F_4(2)$ isomorphic to $C_4(2)$ (which yields a monomial subgroup of $\mathrm{Aut}_Y(B)$) and the fact that $F_4(2)$ has Sylow 3-subgroups of order 3^6. Since $\mathrm{Aut}_G(B)$ has order $2^8 \cdot 3^2$ by assumption that case (b) holds and

$b_0 \neq_Y b_2 b_3^2$ (otherwise s_0 and t_0 would be conjugate in Y against (6.1iii)), we have that $[N(B): N_Y(B)] = 2$. In addition, it is easy to see that the orbits of $N_Y(B)$ on B are those given in case (a) of Proposition 3.1(iii). Hence $b_0^{N_Y(B)} \cup (b_2 b_3^2)^{N_Y(B)} = b_0^{N(B)}$. This implies that $Y = \langle C(\beta)\colon \beta \in b_0^{N(B)} \rangle$ by (6.1i) and so $N(B)$ normalizes Y as claimed. An immediate consequence of this discussion is that $\langle Y, N(B) \rangle$ is a subgroup of $N(Y)$ properly containing Y. As $C(Y) = 1$ by (6.1i), we then have that $N(Y) = \langle Y, \tau \rangle$ where τ is an involution which acts as the graph automorphism on Y.

Let U be a $\langle \tau \rangle$-invariant Sylow 2-subgroup of Y. We argue that U is weakly closed in $U\langle \tau \rangle$ with respect to G. By (2.10ii), U is generated by $t_0^G \cap U$ and $s_0^G \cap U$. Also by (2.10v), every involution in $U\langle \tau \rangle \setminus U$ is conjugate in $\langle Y, \tau \rangle$ to τ, hence has centralizer with order divisible by 13. But $C(s_0)$ and $C(t_0)$ have centralizers with orders prime to 13, hence $U = \langle t_0^G \cap U\langle \tau \rangle, s_0^G \cap U\langle \tau \rangle \rangle$ is weakly closed in $U\langle \tau \rangle$ with respect to G.

We now show that $U\langle \tau \rangle$ is a Sylow 2-subgroup of G. Since $C(t_0) \subseteq Y$ and $C(t_0)$ contains a Sylow 2-subgroup of Y, we must have $C(Z(U)) \subseteq C(t_0^y) \subseteq Y$ for some $y \in Y$. In particular, $U \in \mathrm{Syl}_2(C(Z(U)))$. Also, $|Z(U)| = 4$ by (2.10ii), so $|\mathrm{Aut}_G(Z(U))|_2 \leqslant 2$. As τ acts nontrivially on $Z(U)$, it then follows that $U\langle \tau \rangle \in \mathrm{Syl}_2(N(Z(U)))$. But U is weakly closed in $U\langle \tau \rangle$ with respect to G implies that $N(U\langle \tau \rangle) \subseteq N(Z(U))$. Thus $U\langle \tau \rangle \in \mathrm{Syl}_2(N(U\langle \tau \rangle))$ and we conclude that $U\langle \tau \rangle \in \mathrm{Syl}_2(G)$.

We compute $C(\tau)$. By the assumption that G has characteristic 2-type, $C_Y(\tau) \cong {}^2F_4(2)$ acts faithfully on $P = O_2(C(\tau))$. Since $13 \mid |{}^2F_4(2)|$, $2^{12} < |P/\langle \tau \rangle|$, hence $2^{13} < |P|$. But $|{}^2F_4(2)|_2 = 2^{12}$ and $|F_4(2)|_2 = 2^{24}$. Thus $|G|_2 = 2^{25}$ by the previous paragraph, so $|P| = 2^{13}$ and $PC_Y(\tau)$ contains a Sylow 2-subgroup of G. Let $V \in \mathrm{Syl}_2(C_Y(\tau))$. It is known [15] that $\Omega_1(V) \subset V$, thus $\Omega_1(PV) \subset PV$. On the other hand $PV =_G U\langle \tau \rangle$ and it follows from (2.10ii) that $\Omega_1(U\langle \tau \rangle) = U\langle \tau \rangle$. This leads to a contradiction and the argument that case (b) does not hold is completed.

Finally, we note that $N(B) \subseteq N(Y) = Y$.

We can now complete the proof of Theorem A.

(6.3) $Y = G$, so $G \cong F_4(2)$.

PROOF. Assume that Y is a proper subgroup. We shall apply Holt's theorem to the action of G on the set Ω of left cosets of Y in G. In order to do so, we must, as in §4, verify that t_0 fixes one point of Ω and that $C(t_0)$ contains a Sylow 2-subgroup of G.

We first show that $t_0^G \cap Y = t_0^Y$. Suppose that τ is an involution of Y such that $t_0 = \tau^g$, $g \in G$, but $\tau \neq_Y t_0$. By (2.10iv), $C_Y(\tau)$ contains a Y-conjugate of b_1, so assume, without loss, that $b_1 \in C_Y(\tau)$. Then $b_1^g \in C(t_0)$. Since L contains a Sylow 3-subgroup of $C(t_0) = C_Y(t_0)$ and $b_1^G \cap L = b_1^L$ by (5.1) and (6.2), there exists $y \in C(t_0)$ such that $b^{gy} = b_1$. Thus $\tau^{gy} = t_0$ and $gy \in C(b_1)$. But $C(b_1) \subseteq Y$ as $N(B) \subseteq Y$ and so $\tau \in t_0^Y$ against the choice of τ.

Assume now that $U \in \mathrm{Syl}_2(C(t_0))$. By (2.10ii), $t_0^Y \cap U = \{t_0\}$, hence by the preceding paragraph, $\langle t_0 \rangle$ is strongly closed in $Z(U)$ with respect to G. Then $N(U) \subseteq N(Z(U)) \subseteq C(t_0)$ and we have that $U \in \mathrm{Syl}_2(G)$.

From $t_0^G \cap Y = t_0^Y$ we deduce that t_0 fixes one point of Ω and since t_0 is central in a Sylow 2-subgroup of G, the conditions of Holt's theorem are satisfied. Consideration of the groups given by Holt which have Sylow 3-subgroup of order $3^6 = |G|_3$ leads to a contradiction. Thus $Y = G$ and the result is proved.

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